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The Drinfeld–Sokolov holomorphic bundle and classical W algebras on Riemann surfaces

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Abstract

Developing upon the ideas of an earlier publication it is shown how the theory of classical W algebras can be formulated on a higher genus Riemann surface in the spirit of Krichever and Novikov. The basic geometric object is the Drinfeld–Sokolov principal bundle L associated to a simple complex Lie group G equipped with an $\mathrm{SL}(2,\mathbb{C})$ subgroup S, whose properties are studied in detail. On a multipunctured Riemann surface, the Drinfeld–Sokolov–Krichever–Novikov spaces are defined as a generalization of the customary Krichever–Novikov spaces, their properties are analyzed and standard bases are written down. Finally, a WZWN chiral phase space based on the principal bundle L with a KM type Poisson structure is introduced and, by the usual procedure of imposing first class constraints and gauge fixing, a classical W algebra is produced. The compatibility of the construction with the global geometric data is highlighted.

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1. Introduction

During the last few years, a large body of literature has been devoted to the study of W algebras and to the understanding of their field theoretic realizations. Originally introduced as higher spin extensions of the Virasoro algebra, they were later shown to appear naturally in several contexts, such as cosets of affine Lie algebras, gauged WZWN models, Toda field theory, reductions of the KP hierarchy and, more recently, random matrix models, string theory and 2d quantum gravity (see Ref. [1] for a comprehensive review of the subject and extensive referencing).

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While the local properties of W algebras have been the object of extensive study, a comparatively modest effort has been made in the analysis of their global properties so far [2–5]. The present paper, developing upon and expanding an earlier work [6], is a contribution in such a direction. The approach adopted is inspired on the one hand by the seminal work of Krichever and Novikov [7], which relies on the classical theory of Riemann surfaces and holomorphic bundles thereupon, and on the other by the equally seminal work of Drinfeld and Sokolov [8] and by the techniques of Refs. [9–13], which use the theory of Poisson manifolds and their reductions. Below, I shall provide a brief account of standard results about Toda field theory and W algebras to introduce the basic concepts and motivate the technical analysis presented in later sections (see also Ref. [14] for a review).

The Toda field equations can be put in the form of a zero curvature condition for a connection $\mathscr{C}$ satisfying a certain grading constraint. This allows for the integrability of the Toda equations, a well established result [15]. It also hints to its geometrical nature, which indeed is describable in the language of the theory of holomorphic principal bundles.

The basic algebraic structure of Toda equations is a simple complex Lie group G with an $SL(2,\mathbb{C})$ subgroup S with Lie algebras $\mathfrak{g}$ and $\mathfrak{s}$, respectively. $\mathfrak{g}$ is equipped with a conjugation $\dagger$ corresponding to a compact antiinvolution of $\mathfrak{g}$ and leaving $\mathfrak{s}$ invariant. $\mathfrak{s}$ has standard generators t_{-1}, t_0, t_{+1} satisfying $[t_{+1}, t_{-1}]=2t_0$, $[t_0, t_{\pm 1}]=\pm t_{\pm 1}$ and $t_a^\dagger=t_{-a}$. To t_0, there is associated a half-integer gradation of $\mathfrak{g}$.

On a Riemann surface Σ of higher genus with holomorphic canonical line bundle k, one can define a holomorphic G-bundle L^0, called the Drinfeld–Sokolov bundle in Ref. [4], by

$$L^0{}_{ab}=k^{-t_0}{}_{ab}\,, \tag{1.1}$$

where a and b are coordinate labels.

The Toda field equations on the Riemann surface Σ are the zero curvature condition for the connection $\mathscr{C}=\mathrm{d}z\,C+\mathrm{d}\bar{z}\,C^*$ of L^0 given by

$$C=\partial e^{\phi}e^{-\phi}-\partial\ln g t_0+\tfrac{1}{2}t_{+1}\,, \tag{1.2a}$$

$$C^*=2e^{ad\,\phi}t_{-1}g\,, \tag{1.2b}$$

where the Toda field ϕ is a section of $\mathrm{Ad}\,L^0$ such that $\phi^\dagger=\phi$ and $[t_0,\phi]=0$ and g is a metric of Σ compatible with its holomorphic structure. Explicitly,

$$\bar{\partial}(\partial e^{\phi}e^{-\phi})-\bar{\partial}\partial\ln g t_0+[t_{+1}, e^{ad\,\phi}t_{-1}]g=0\,. \tag{1.3}$$

As shown in Ref. [5], this is just Hitchin's self-duality equation for the Higgs pair (L^0,Ω), where $\Omega=\frac{1}{2}t_{+1}$ and the unitary connection is that of the Hermitian metric of L^0 given by $e^{\phi}g^{-t_0}$.

Let G_- be the negative graded subgroup of G. One can show that, on any coordinate path, there exists a G_- valued smooth solution γ of the equation

$$\gamma^{-1}\bar\partial\gamma+2e^{ad\,\phi}t_{-1}g=0\,, \tag{1.4}$$

such that on two overlapping coordinate domains

$$\gamma_a=L_{ab}\gamma_b L^0{}_{ab}{}^{-1}\,, \tag{1.5}$$

where L is the holomorphic G-bundle defined by

$$L_{ab}=L^0{}_{ab}\exp(\partial_a k_{ab}{}^{-1}t_{-1})\,. \tag{1.6}$$

A proof of this theorem for $G=\mathrm{SL}(n,\mathbb{C})$ was given in Ref. [5] but the result holds in general. The integrability of Eq. (1.4) requires crucially the use of Toda equation (1.3). The solution is however non unique. L was called Drinfeld–Sokolov bundle in Ref. [6]. In fact, L^0 and L are *distinct* holomorphic forms of the same smooth G-bundle. However, while L^0 has no flat form, L does. Hence, L admits a *holomorphic* connection.

A holomorphic connection $\mathcal{J}$ of the bundle L can be obtained directly from the Toda connection $\mathcal{C}$ by a "gauge transformation" γ satisfying (1.4). Eq. (1.4) is indeed equivalent to the vanishing of (0, 1) component of $\mathcal{J}=\mathrm{d}z\,J+\mathrm{d}\bar z\,J^*$:

$$J=\mathrm{Ad}\,\gamma C+\partial\gamma\gamma^{-1}=\mathrm{Ad}\,\gamma(\partial e^{\phi}e^{-\phi}-\partial\ln g t_0+\tfrac{1}{2}t_{+1})+\partial\gamma\gamma^{-1}\,, \tag{1.7a}$$

$$J^*=\mathrm{Ad}\,\gamma C^*+\bar\partial\gamma\gamma^{-1}=0\,. \tag{1.7b}$$

The zero curvature condition, equivalent to Toda equations, now reads simply

$$\bar\partial J=0\,. \tag{1.8}$$

In fact, up to a factor κ, J is the WZWN current and is given by

$$J=\partial h h^{-1}\,,\qquad h=\gamma e^{\phi}g^{-t_0}S\gamma'^{\dagger}S^{-1}\,, \tag{1.9}$$

where γ' is a solution of (1.4), not necessarily equal to γ, and $S=e^{i\pi t_0}$. Since the G-bundle L has a large holomorphic gauge group, it is possible to choose γ in such a way that the current J in (1.7a) is of the form

$$J=\tfrac{1}{2}t_{+1}-Rt_{-1}+\kappa^{-1}W\,,\quad \mathrm{ad}\,t_{-1}W=0\,, \tag{1.10}$$

where R is a background holomorphic projective connection.

The above discussion shows that, in the present geometrical setting, the space of chiral WZWN currents is to be identified with the affine space of holomorphic connections of the holomorphic G-bundle L. The chiral currents belonging to Toda field theory span a subspace of such space, which is, up to holomorphic gauge equivalence, the one defined by the constraint (1.10).

The canonical Poisson structure of Toda field theory induces a Poisson structure on the space of the Toda connection J of the form (1.10), which, as a consequence, obeys a classical W algebra [16]. In Refs. [9,10], it was shown that Toda field theory can be formulated as a conformally invariant Hamiltonian reduction of WZWN theory and that the classical W algebra structure can be recovered in this way. This is also the point of view adopted in this paper.

Following Ref. [7], the twice punctured Riemann surface $\Sigma \backslash \{P_-, P_+\}$ obtained from Σ by removing two points P_- and P_+ in general position is considered, generalizing the customary cylindrical setting. The appropriate WZWN phase space consists in the affine space of meromorphic connections of the bundle L holomorphic off P_- and P_+ equipped with a suitable Poisson structure of Kac–Moody type. Then, following Ref. [10], the WZWN phase space is reduced by imposing first class constraints compatible with the conformal symmetry and gauge fixing. A classical W algebra is yielded in this way.

The plan of the paper is as follows. In Section 2, a brief account of the basic properties of $\mathfrak{sl}(2, \mathbb{C})$ embeddings into simple complex Lie algebras used in the sequel is given. In Section 3, a systematic study of the Drinfeld–Sokolov bundle is carried out. Section 4 contains the basic notions of Krichever–Novikov theory and the illustration of their generalization in the present context. Finally, in Sections 5, 6 and 7, the theory of the WZWN phase space and its reduction is presented, the properties of the reduced phase space are studied and the emergence of a classical W algebra is shown.

2. $\mathfrak{sl}(2, \mathbb{C})$ embeddings into simple complex Lie algebras

In this section, I shall briefly expound the main results on the theory of $\mathfrak{sl}(2, \mathbb{C})$ embeddings into simple complex Lie algebras which will be frequently relied upon in the following. A classic treatment of the subject is provided by Ref. [17].

Remark. In this section, $\mathfrak{g}$ is a simple complex Lie algebra. $\mathfrak{s}$ is an $\mathfrak{sl}(2, \mathbb{C})$ subalgebra of $\mathfrak{g}$. $\mathfrak{c}_\mathfrak{s}$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$.

Theorem 2.1. *$\mathfrak{g}$ is completely reducible under ad $\mathfrak{s}$.*

Proof. In fact, since $\mathfrak{s}$ is a simple algebra, $\mathfrak{g}$ is completely reducible under ad $\mathfrak{s}$, by Weyl's theorem (th. 8, ch. III of Ref. [18]). □

The non triviality of $\mathfrak{c}_\mathfrak{s}$ measures the degeneracy of the spectrum of $\mathfrak{sl}(2, \mathbb{C})$ irreducible representations in the reduction.

Let us denote by Π the set of the representations of $\mathfrak{sl}(2, \mathbb{C})$ appearing in the reduction of $\mathfrak{g}$ by ad $\mathfrak{s}$, each counted with its multiplicity, by $j_\eta \in \mathbb{Z}/2$ the spin of a representation $\eta \in \Pi$ and by I_η the set $\{m \,|\, m \in \mathbb{Z}/2,\ |m| \leqslant j_\eta,\ j_\eta - m \in \mathbb{Z}\}$. Let us further set $j_* = \max\{j_\eta \,|\, \eta \in \Pi\}$. Since ad $\mathfrak{s}$ acts irreducibly on $\mathfrak{s}$, there is a distinguished representation Π corresponding to $\mathfrak{s}$, which will be denoted by o.

Theorem 2.2. *$\mathfrak{s}$ has a set of generators t_d, $d = -1, 0, +1$, satisfying the relations*

$$[t_{+1}, t_{-1}] = 2t_0 , \qquad [t_0, t_{\pm 1}] = \pm t_{\pm 1} . \tag{2.1}$$

Associated to these, there is a set $\{t_{\eta,m} | \eta \in \Pi, m \in I_\eta\}$ *of generators of* $\mathfrak{g}$ *such that*

$$[t_d, t_{\eta,m}] = C^d_{j_\eta,m} t_{\eta,m+d} , \quad d = -1, 0, +1 , \tag{2.2a}$$

$$C^{\pm 1}_{j,m} = [j(j+1) - m(m \pm 1)]^{1/2} , \qquad C^0_{j,m} = m . \tag{2.2b}$$

The Lie brackets of the $t_{\eta,m}$ *have the following form:*

$$[t_{\eta,m}, t_{\zeta,n}] = \sum_{\xi \in \Pi, k \in I_\xi} F_{\eta,\zeta}{}^{\xi} (j_\eta, m; j_\zeta, n | j_\xi, k) t_{\xi,k} , \tag{2.3}$$

where $(j_1, m_1; j_2, m_2 | j_3, m_3)$ *is a Clebsch–Gordan coefficient and the* $F_{\eta,\zeta}{}^{\xi}$ *are constants depending only on the* $\mathfrak{sl}(2, \mathbb{C})$ *embedding* $\mathfrak{s}$ *and enjoying the following properties.* $F_{\xi,\eta}{}^{\zeta}$ *vanishes unless* $|j_\xi - j_\eta| \leqslant j_\zeta \leqslant j_\xi + j_\eta$ *and* $j_\xi + j_\eta - j_\zeta \in \mathbb{Z}$. *Further, for any* $\xi, \eta, \zeta \in \Pi$,

$$F_{\xi,\eta}{}^{\zeta} = -(-1)^{j_\xi + j_\eta - j_\zeta} F_{\eta,\xi}{}^{\zeta} , \tag{2.4}$$

and, for any $\xi, \eta, \zeta, \lambda \in \Pi$ *and any* $j \in \mathbb{Z}/2$, $j \geqslant 0$ *with* $|j_\xi - j_\eta| \leqslant j \leqslant j_\xi + j_\eta$ *and* $j_\xi + j_\eta - j \in \mathbb{Z}$,

$$\sum_{\mu \in \Pi} \{ F_{\xi,\eta}{}^{\mu} F_{\mu,\zeta}{}^{\lambda} \delta_{j_\mu, j} + F_{\zeta,\xi}{}^{\mu} F_{\mu,\eta}{}^{\lambda} \Omega(j_\xi, j_\eta, j_\zeta, j_\lambda; j, j_\mu)$$
$$+ F_{\eta,\zeta}{}^{\mu} F_{\mu,\xi}{}^{\lambda} \Omega(j_\eta, j_\zeta, j_\xi, j_\lambda; j_\mu, j) \} = 0 , \tag{2.5}$$

where $\Omega(j_1, j_2, j_3, j_4; j_6, j_6) = (-1)^{j_3 + j_5 - j_4} (2j_5 + 1)^{1/2} (2j_6 + 1)^{1/2} W(j_1, j_2, j_3, j_4; j_5, j_6)$ *and* $W(j_1, j_2, j_3, j_4; j_5, j_6)$ *is a Racah–Wigner function [19]. Finally, one has*

$$t_{o,\pm 1} = \mp 2^{-1/2} t_{\pm 1} , \qquad t_{o,0} = t_0 , \tag{2.6}$$

$$F_{o,\eta}{}^{\zeta} = F_{\eta,o}{}^{\zeta} = -\delta_{\eta,\zeta} [j_\eta (j_\eta + 1)]^{1/2} . \tag{2.7}$$

Proof. (2.1) and (2.2) are standard results from the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ [17,18]. Let $[t_{\eta,m}, t_{\zeta,n}] = \sum_{\xi \in \Pi, k \in I_\xi} F_{\eta,m;\zeta,n}{}^{\xi,k} t_{\xi,k}$, where the $F_{\eta,m;\zeta,n}{}^{\xi,k}$ are structure constants. From the Jacobi identity for the triple of generators t_d, $t_{\eta,m}$, $t_{\zeta,n}$, one gets

$$C^{-d}_{j_\xi,k} F_{\eta,m;\zeta,n}{}^{\xi,k-d} - C^d_{j_\eta,m} F_{\eta,m+d;\zeta,n}{}^{\xi,k} - C^d_{j_\zeta,n} F_{\eta,m;\zeta,n+d}{}^{\xi,k} = 0 ,$$
$$d = -1, 0, +1 . \tag{2.8}$$

For fixed $\xi, \eta, \zeta \in \Pi$, such relations have the same form as the recurrence relation of the Clebsch–Gordan coefficients. This yields (2.3). Eqs. (2.4) and (2.5) follow from the antisymmetry and the Jacobi identity of the Lie brackets and from well-known properties of the Clebsch–Gordan coefficients [19]:

$$(j_2, m_2; j_1, m_1 | j_3, m_3) = (-1)^{j_1 + j_2 - j_3} (j_1, m_1; j_2, m_2 | j_3, m_3) ;$$

$$\begin{aligned}&(j_1, m_1; j_2, m_2 | j_{12}, m_1+m_2)(j_{12}, m_1+m_2; j_3, m_3 | j_4, m_4)\\&\quad = \sum_{j_{23}} (j_2, m_2; j_3, m_3 | j_{23}, m_2+m_3)(j_1, m_1; j_{23}, m_2+m_3 | j_4, m_4)\\&\qquad \times (2j_{12}+1)^{1/2}(2j_{23}+1)^{1/2} W(j_1, j_2, j_4, j_3; j_{12}, j_{23}) .\end{aligned}$$

Eq. (2.6) follows from comparing (2.1) and (2.2). Eq. (2.7) follows from (2.6) and from comparing (2.3) and (2.2). □

Denote by $(\cdot, \cdot)$ the Cartan–Killing form of $\mathfrak{g}$.

Theorem 2.3. *One has*

$$(t_{+1}, t_{-1}) = 2(t_0, t_0) . \tag{2.9}$$

For each representation $\eta \in \Pi$, there is a conjugate representation $\bar\eta$ such that $j_\eta = j_{\bar\eta}$. Further $\bar{\bar\eta} = \eta$ and $\bar\eta = \eta$ if and only if $j_\eta \in \mathbb{Z}$. Moreover, for η, $\zeta \in \Pi$, $m \in I_\eta$ and $n \in I_\zeta$

$$(t_{\eta,m}, t_{\zeta,n}) = N_\eta \delta_{\eta,\bar\zeta} (-1)^{j_\eta - m} \delta_{m,-n} , \tag{2.10a}$$

where N_η is a normalization constant such that

$$N_{\bar\eta} = (-1)^{2j_\eta} N_\eta . \tag{2.10b}$$

In particular, one has $o = \bar o$ and

$$N_o = -(t_0, t_0) . \tag{2.11}$$

Proof. Eq. (2.9) follows easily from (2.1) and the ad invariance of the Cartan–Killing form. For any homogeneous polynomial P of degree p in ad t_d, $d = -1, 0, +1$, one has $(Px, y) + (-1)^p (x, Py) = 0$ for all x, $y \in \mathfrak{g}$. Choosing $P = \operatorname{ad} t_0$, $\frac{1}{2}(\operatorname{ad} t_{-1} \operatorname{ad} t_{+1} + \operatorname{ad} t_{+1} \operatorname{ad} t_{-1}) + (\operatorname{ad} t_0)^2$ and $x = t_{\eta,m}$ and $y = t_{\zeta,n}$, one finds

$$(t_{\eta,m}, t_{\zeta,n}) = N_{\eta,\zeta,m} \delta_{j_\eta, j_\zeta} \delta_{m,-n} . \tag{2.12}$$

Choosing for $P = \operatorname{ad} t_{+1}$ and $x = t_{\eta,m-1}$ and $y = t_{\zeta,-m}$ and using (2.12), one finds further

$$N_{\eta,\zeta,m} = N_{\eta,\zeta} (-1)^{j_\eta - m} . \tag{2.13}$$

The non singularity of the Cartan–Killing form implies that the matrix $N_{\eta,\zeta}$ is non singular. From (2.12) and (2.13), it follows that

$$N_{\eta,\zeta} = (-1)^{2j_\eta} N_{\zeta,\eta} , \tag{2.14}$$

where $j_\eta = j_\zeta$. Hence, the matrix $N_{\eta,\zeta}$ is either symmetric or antisymmetric. By a congruence, it can be put in the form of either diagonal matrix with non zero diagonal entries or a direct sum of matrices of the form $i\sigma_2$ with a non zero coefficient, respectively, where σ_2 is a Pauli matrix. In both cases, for each $\eta \in \Pi$ there exists a unique $\bar\eta \in \Pi$ such that $N_{\eta,\bar\eta} \neq 0$. From here, (2.10) follows easily. The remaining statements are obvious. □

Associated to the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, there is a half-integer grading of $\mathfrak{g}$. For any $m\in\mathbb{Z}/2$, one sets $\mathfrak{g}_m=\oplus_{\eta\in\Pi}\mathbb{C}t_{\eta,m}$. This is just the eigenspace of $\mathrm{ad}\, t_0$ with eigenvalue m. Note that, $\mathfrak{g}_m=0$ for $|m|>j_*$. One also introduces the subspaces $\mathfrak{g}_{<m}=\oplus_{k<m}\mathfrak{g}_k$, $\mathfrak{g}_{\neq m}=\oplus_{k\neq m}\mathfrak{g}_k$, etc.

It is readily seen that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. For any $m\in\mathbb{Z}/2$ with $m>0$, $\mathfrak{g}_{<-m}$ and $\mathfrak{g}_{>m}$ are nilpotent subalgebras of $\mathfrak{g}$. It can be verified that $\mathfrak{c}_\mathfrak{s}=\oplus_{\eta\in\Pi, j_\eta=0}\mathbb{C}t_{\eta,0}$. In particular, $\mathfrak{c}_\mathfrak{s}$ is a subalgebra of $\mathfrak{g}_0$. One also has the identity $\ker \mathrm{ad}\, t_{\pm 1}=\oplus_{\eta\in\Pi}\mathbb{C}t_{\eta,\pm j_\eta}$.

For principal $\mathfrak{sl}(2,\mathbb{C})$ embeddings, $\mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Further, $\mathfrak{c}_\mathfrak{s}$ is trivial, Π contains only integer spin representations of strictly positive spin with unit multiplicity. This is no longer true for non principal $\mathfrak{sl}(2,\mathbb{C})$ embeddings.

3. The Drinfeld–Sokolov holomorphic G-bundle and its properties

This section is dedicated to the study of the main properties of the Drinfeld–Sokolov (DS) bundle, which is the basic geometric object entering in the construction of classical W algebras illustrated in Sections 5, 6 and 7. The analysis developed below envisages only the local properties of $\mathrm{SL}(2,\mathbb{C})$ embeddings into simple complex Lie groups and, thus, is amenable by the Lie algebraic methods developed in Section 2.

Remark. Throughout this section, the following assumptions are made. Σ is a compact connected Riemann surface without boundary of genus $l\geqslant 2$. $k^{\otimes 1/2}$ is a fixed theta characteristic. h is a fixed element of $\mathbb{Z}/2$. G is a connected simple complex Lie group. S is an $\mathrm{SL}(2,\mathbb{C})$ subgroup of G.

Recall that $k^{\otimes 1/2\otimes 2}=k$, where k is the holomorphic canonical 1-cocycle of Σ defined by $k_{ab}=\partial_a z_b$.

I denote by z the generic holomorphic coordinate of Σ and by ∂ the operator $\partial/\partial z$. I further use lower Latin indices $a, b, c, \ldots$ as labels for different coordinates. Further, $k^{\otimes h}$ is short for $k^{\otimes 1/2\otimes 2h}$. For any holomorphic 1-cocycle K on Σ representing some holomorphic bundle on Σ and any one empty open set U of Σ, I denote by $\Gamma(U, \mathcal{O}(K))$ and $\Gamma(U, \mathcal{M}(K))$ the spaces of holomorphic and meromorphic sections of K on U, respectively. Finally, I denote by exp the exponential map of G and by C_S the centralizer of S in G.

3.1. The DS holomorphic G-bundle L

Definition 3.1. Let t_{-1}, t_0, t_{+1} be the standard generators of $\mathfrak{s}$. For any two overlapping coordinate domains, one sets

$$L_{ab}=\exp(-2\ln k^{\otimes 1/2}{}_{ab}t_0)\exp(\partial_a k_{ab}{}^{-1}t_{-1}), \tag{3.1}$$

where exp is the exponential map of G.

Theorem 3.1. *$L=\{L_{ab}\}$ is a holomorphic G-valued 1-cocycle on Σ. It thus defines a holomorphic G-bundle canonically associated to the pair (G,S).*

Proof. One has $\exp(4\pi i t_0)=1$. Further, $k^{\otimes 1/2}$ is a holomorphic 1-cocycle on Σ. From these facts, it is easily checked that $\{\exp(-2\ln k^{\otimes 1/2}{}_{ab}t_0)\}$ is a holomorphic G-valued 1-cocycle on Σ. Using (2.1), it is then straightforward to verify that $\{L_{ab}\}$, also, is a holomorphic G-valued 1-cocycle on Σ. □

L will be called the DS bundle [4,6,20]. In application to classical W algebras, the relevant 1-cocycles are of the form $k^{\otimes h}\otimes \mathrm{Ad}\,L$, where $h\in\mathbb{Z}/2$. Below, I shall carry out a systematic study of them.

3.2. Generalities on $\Gamma(\Sigma,\mathcal{M}(k^{\otimes h}\otimes \mathrm{Ad}\,L))$

Let $\Phi\in\Gamma(\Sigma,\mathcal{M}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$. Φ can be expanded in the basis $\{t_{\eta,m}\,|\,\eta\in\Pi,\ m\in I_\eta\}$ of $\mathfrak{g}$ canonically associated to its $\mathfrak{sl}(2,\mathbb{C})$ subalgebra $\mathfrak{s}$, obtaining

$$\Phi_a=\sum_{\eta\in\Pi,m\in I_\eta}\Phi_{\eta,m\,a}t_{\eta,m}, \tag{3.2}$$

where the $\Phi_{\eta,m\,a}$ are certain meromorphic functions.

Theorem 3.2. *For any $\Phi\in\Gamma(\Sigma,\mathcal{M}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$, one has*

$$\Phi_{\eta,m\,a}=k^{\otimes h}{}_{ab}\sum_{n\in I_\eta}L^{(\eta)}{}_{ab\,m}{}^{n}\Phi_{\eta,n\,b}, \tag{3.3}$$

where $L^{(\eta)}=\{L^{(\eta)}{}_{ab}\}$ is the holomorphic $\mathrm{SL}(2j_\eta+1,\mathbb{C})$-valued 1-cocycle

$$L^{(\eta)}{}_{ab\,m}{}^{n}=\frac{1}{(n-m)!}\left(\prod_{r\in\mathbb{N},1\leqslant r\leqslant n-m}C^{+1}_{j_\eta,n-r}\right)k^{\otimes -m}{}_{ab}(\partial_a k_{ab}{}^{-1})^{n-m},$$

$$m,n\in I_\eta,\ m\leqslant n, \tag{3.4a}$$

$$L^{(\eta)}{}_{ab\,m}{}^{n}=0,\quad m,n\in I_\eta,\ m>n, \tag{3.4b}$$

where $C^d_{j,k}$ is given by (2.2b).

Proof. This follows easily from substituting the expansion (3.2) into the relation $\Phi_a=k^{\otimes h}{}_{ab}\,\mathrm{Ad}\,L_{ab}\Phi_b$ and then using (2.2). The calculation is straightforward. □

The following technical theorem will be of crucial importance in the following treatment. Recall that a projective connection R is a holomorphic 0-cochain $\{R_a\}$ such that $R_a = k_{ab}{}^2(R_b - \{z_a, z_b\})$, where $\{z_a, z_b\} = \partial_b^2 \ln \partial_b z_a - \frac{1}{2}(\partial_b \ln \partial_b z_a)^2$ is the Schwarzian.

Theorem 3.3. *Let R be a holomorphic projective connection. Let $\eta \in \Pi$ and $\mu \in I_\eta$ with either $\mu < h$ or $\mu \geqslant j_\eta + 2h$. Let $\phi \in \Gamma(\Sigma, \mathcal{M}(k^{\otimes h-\mu}))$. Then, there exists a unique element $\Phi \in \Gamma(\Sigma, \mathcal{M}(k^{\otimes h} \otimes \mathrm{Ad}\, L))$ such that*

$$\Phi_{\eta,m} = \phi \delta_{\mu,m}\,, \quad m \in I_\eta,\, m \geqslant \mu\,, \tag{3.5a}$$

$$\Phi_{\eta,m} = \frac{C^{+1}_{j_\eta,m}}{g_{\mu-h,m-h}}\,(\partial \Phi_{\eta,m+1} + C^{+1}_{j_\eta,m+1} R \Phi_{\eta,m+2})\,, \quad m \in I_\eta,\, m < \mu\,, \tag{3.5b}$$

$$\Phi_{\zeta,n} = 0\,, \quad \zeta \in \Pi,\, \zeta \neq \eta,\, n \in I_\zeta\,, \tag{3.5c}$$

where $g_{x,y} = \frac{1}{2}(x(x+1) - y(y+1))$. Φ depends linearly on ϕ. Moreover, if $\phi \in \Gamma(\Sigma, \mathcal{O}(k^{\otimes h-\mu}))$, then $\Phi \in \Gamma(\Sigma, \mathcal{O}(k^{\otimes h} \otimes \mathrm{Ad}\, L))$.

Proof. $g_{\mu-h,m-h}$ vanishes for $m = \mu$, $-\mu + 2h - 1$. So, $g_{\mu-h,m-h}$ will vanish for some $m \in I_\eta$ with $m < \mu$ if $-j_\eta \leqslant -\mu + 2h - 1 < \mu$, i.e., $-\frac{1}{2} + h < \mu \leqslant j_\eta + 2h - 1$. The latter relation, however, cannot be fulfilled by the assumptions made on μ. Hence, $g_{\mu-h,m-h} \neq 0$ in the range of m values indicated. Further, (3.5b) provides a recurrence relation for the components $\Phi_{\eta,m}$ with $m \leqslant \mu$ with (3.5a) as initial condition. This allows the unique determination of all $\Phi_{\eta,m}$ in terms of ϕ, R and their derivatives. It is also apparent that $\Phi_{\eta,m}$ is meromorphic. To complete the proof, one has only to show that the $\Phi_{\eta,m}$, as determined by (3.5b), glue according to (3.3) with $L^{(\eta)}$ given by (3.4). The verification is trivial for $m \geqslant \mu$. To show that the statement is true also for $m < \mu$, one proceeds by induction. Suppose that one has been able to show that the $\Phi_{\eta,n}$ glue according to (3.3) for $n \in I_\eta$ with $m \leqslant n$, where $m \in I_\eta$ with $-j_\eta < m \leqslant \mu$. By using this information, let us show that $\Phi_{\eta,m-1}$, also, glues according to (3.3). Now, since $m - 1 \in I_\eta$ and $m - 1 < \mu$, one can use (3.5b). Using the inductive hypothesis and (3.5a) one computes

$$\begin{aligned}
\Phi_{\eta,m-1\,a} = \frac{C^{+1}_{j_\eta,m-1}}{g_{\mu-h,m-1-h}}\, k^{\otimes 1+h}{}_{ab} \Bigg\{ & -h \partial_a k_{ab}{}^{-1} \sum_{n \in I_\eta, m \leqslant n \leqslant \mu} L^{(\eta)}{}_{ab\,m}{}^n \Phi_{\eta,n\,b} \\
& + \sum_{n \in I_\eta, m \leqslant n \leqslant \mu} \partial_b L^{(\eta)}{}_{ab\,m}{}^n \Phi_{\eta,n\,b} \\
& + \sum_{n \in I_\eta, m-1 \leqslant n \leqslant \mu - 1} \left(\frac{g_{\mu-h,n-h}}{C^{+1}_{j_\eta,n}} \right) L^{(\eta)}{}_{ab\,m}{}^{n+1} \Phi_{\eta,n\,b} \\
& - R_b \sum_{n \in I_\eta, m+1 \leqslant n \leqslant \mu} C^{+1}_{j_\eta,n-1} L^{(\eta)}{}_{ab\,m}{}^{n-1} \Phi_{\eta,n\,b} \\
& + C^{+1}_{j_\eta,m} k_{ab} (R_b - \{z_a, z_b\}) \sum_{n \in I_\eta, m+1 \leqslant n \leqslant \mu} L^{(\eta)}{}_{ab\,m+1}{}^n \Phi_{\eta,n\,b} \Bigg\}\,.
\end{aligned} \tag{3.6}$$

Let us compute $\partial_b L^{(\eta)}{}_{ab\,m}{}^n$. To this end, one uses (3.4) and the identities $k_{ab}{}^{-1}\partial_a{}^2 k_{ab}{}^{-1} - \frac{1}{2}(\partial_a k_{ab}{}^{-1})^2 = \{z_a, z_b\}$ and $(C^{+1}_{j,i-1})^2 - (C^{+1}_{j,l})^2 = (l+i)(l-i+1)$, which follow easily from the definition of the Schwarzian and from (2.2b), respectively. One finds

$$\partial_b L^{(\eta)}{}_{ab\,i}{}^l = -(C^{+1}_{j_\eta,l}/2) L^{(\eta)}{}_{ab\,i}{}^{l+1} + (C^{+1}_{j_\eta,i-1}/2) k_{ab}{}^{-1} L^{(\eta)}{}_{ab\,i-1}{}^l$$
$$+ C^{+1}_{j_\eta,i} k_{ab}\{z_a, z_b\} L^{(\eta)}{}_{ab\,i+1}{}^l, \quad i, l \in I_\eta, i \leqslant l, \tag{3.7}$$

where the first term vanishes for $l = j_\eta$. From (3.4), it is straightforward to verify also the identities

$$C^{+1}_{j_\eta,i} k_{ab} L^{(\eta)}{}_{ab\,i+1}{}^l = C^{+1}_{j_\eta,l-1} L^{(\eta)}{}_{ab\,i}{}^{l-1}, \quad i, l \in I_\eta, i+1 \leqslant l, \tag{3.8}$$

$$C^{+1}_{j_\eta,i-1} \partial_a k_{ab}{}^{-1} L^{(\eta)}{}_{ab\,i}{}^l = (l-i+1) k_{ab}{}^{-1} L^{(\eta)}{}_{ab\,i-1}{}^l, \quad i, l \in I_\eta, i \leqslant l. \tag{3.9}$$

Using (3.7)-(3.9) in (3.6) and performing some simplifications, one obtains

$$\Phi_{\eta,m-1\,a} = \frac{C^{+1}_{j_\eta,m-1}}{g_{\mu-h,m-1-h}} k^{\otimes 1+h}{}_{ab} \Bigg\{ \sum_{n\in I_\eta, m\leqslant n\leqslant\mu} \Bigg[-\tfrac{1}{2} C^{+1}_{j_\eta,n} L^{(\eta)}{}_{ab\,m}{}^{n+1}$$
$$+ \left(-h \frac{n-m+1}{C^{+1}_{j_\eta,m-1}} + \tfrac{1}{2} C^{+1}_{j_\eta,m-1} \right) k_{ab}{}^{-1} L^{(\eta)}{}_{ab\,m-1}{}^n \Bigg] \Phi_{\eta,n\,b}$$
$$+ \sum_{n\in I_\eta, m-1\leqslant n\leqslant\mu-1} \left(\frac{g_{\mu-h,n-h}}{C^{+1}_{j_\eta,n}} \right) L^{(\eta)}{}_{ab\,m}{}^{n+1} \Phi_{\eta,n\,b} \Bigg\}. \tag{3.10}$$

Employing (3.8) to express $L^{(\eta)}{}_{ab\,m}{}^{n+1}$ in terms of $L^{(\eta)}{}_{ab\,m-1}{}^n$ one gets, after a little algebra,

$$\Phi_{\eta,m-1\,a} = k^{\otimes h}{}_{ab} \sum_{n\in I_\eta, m-1\leqslant n\leqslant\mu} L^{(\eta)}{}_{ab\,m-1}{}^n \Phi_{\eta,n\,b}. \tag{3.11}$$

By induction the proof is completed. The remaining statements are obvious. □

Definition 3.2. For R, η, μ and ϕ as in Thm. 3.3, let $F_{h,\eta,\mu}(\phi|R)$ be the unique element of $\Phi \in \Gamma(\Sigma, \mathcal{M}(k^{\otimes h} \otimes \mathrm{Ad}\, L))$ satisfying (3.5).

By explicit calculation, one finds

$$F_{h,\eta,\mu}(\phi|R)_{\eta,m} = \phi\delta_{\mu,m}, \quad m \in I_\eta, m \geqslant \mu, \tag{3.12a}$$

$$F_{h,\eta,\mu}(\phi|R)_{\eta,m} = N_{h,\eta,\mu,m} D_{h,\mu,\mu-m}(R)\phi, \quad m \in I_\eta, m < \mu, \tag{3.12b}$$

where

$$N_{h,\eta,\mu,m} = \prod_{n\in I_\eta, m\leqslant n\leqslant\mu-1} \frac{C^{+1}_{j_\eta,n}}{g_{\mu-h,n-h}}, \tag{3.12c}$$

$$D_{h,\mu,1}(R)=\partial\ ,$$

$$D_{h,\mu,2}(R)=\partial^2+g_{\mu-h,\mu-h-1}R\ ,$$

$$D_{h,\mu,3}(R)=\partial^3+(g_{\mu-h,\mu-h-1}+g_{\mu-h,\mu-h-2})R\partial+g_{\mu-h,\mu-h-1}(\partial R)\ ,$$

$$\begin{aligned}D_{h,\mu,4}(R)=\partial^4&+(g_{\mu-h,\mu-h-1}+g_{\mu-h,\mu-h-2}+g_{\mu-h,\mu-h-3})R\partial^2\\&+(2g_{\mu-h,\mu-h-1}+g_{\mu-h,\mu-h-2})(\partial R)\partial\\&+g_{\mu-h,\mu-h-1}((\partial^2R)+g_{\mu-h,\mu-h-3}R^2)\ ,\end{aligned}$$

etc. (3.12d)

These operators provide non trivial generalizations of the standard Bol operators [21].

From the above, one deduces the following theorem. For any $w\in\mathbb{Z}/2$ such that $w\geqslant 0$, let $J(k^{\otimes -w})$ denote the $2w$th jet extension of $k^{\otimes -w}$, i.e., the holomorphic $\mathrm{SL}(2w+1,\ \mathbb{C})$-valued 1-cocycle defined by $\partial_a{}^m\phi_a=\sum_{n=0}^{2w}J(k^{\otimes -w})_{abm}{}^n\partial_b{}^n\phi_b$, $m=0,1,\dots,2w$ for any $\phi\in\Gamma(\Sigma,\mathscr{M}(k^{\otimes -w}))$.

Theorem 3.4. *One has the direct sum decomposition*

$$\mathrm{Ad}\,L\cong\bigoplus_{\eta\in\Pi}L^{(\eta)}\ . \tag{3.13}$$

Further, for any $\eta\in\Pi$, one has the holomorphic equivalence

$$L^{(\eta)}\cong J(k^{\otimes -j_\eta})\ . \tag{3.14}$$

Proof. Eq. (3.13) follows from (3.3) and (3.4) directly. Choose a holomorphic projective connection R. From (3.5), it is easily verified that, for any $\eta\in\Pi$, $\phi\in\Gamma(\Sigma,\mathscr{M}(k^{\otimes -j_\eta}))$ and $m\in I_\eta$,

$$F_{0,\eta,j_\eta}(\phi|R)_{\eta,m}=\sum_{n=0}^{2j_\eta}\mathscr{F}^{(j_\eta)}(R)_m{}^n\partial^n\phi\ , \tag{3.15}$$

where, for $w\in\mathbb{Z}/2$ with $w\geqslant 0$, $\mathscr{F}^{(w)}(R)$ is a $(2w+1)\times(2w+1)$ invertible lower triangular matrix whose entries are differential polynomials in R. From (3.15), ϕ being arbitrary, (3.14) follows. □

3.3. Study of $\Gamma(\Sigma,\ \mathscr{O}(k^{\otimes h}\otimes \mathrm{Ad}\,L))$

Theorem 3.5. *Let $\Phi\in\Gamma(\Sigma,\ \mathscr{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$. Then, in the expansion (3.2), one has*

$$\Phi_{\eta,m}=0\ ,\quad \eta\in\Pi,\ m\in I_\eta,\ m>h\ , \tag{3.16a}$$

$$\Phi_{\eta,h}=0\ ,\quad \eta\in\Pi,\ h\in I_\eta,\ j_\eta>-h\ . \tag{3.16b}$$

Proof. From (3.3) and(3.4), it follows that, if $\Phi_{\eta,m}=0$ for $m\in I_\eta$, $m>n$ with $n\in I_\eta$, then $\Phi_{\eta,n}\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h-n}))$. On the other hand, $\Gamma(\Sigma, \mathcal{O}(k^{\otimes h-n}))=0$ if $h<n$, from the Riemann–Roch theory [22]. From these properties, beginning with $m=j_\eta$ and proceeding by induction, one can easily verify (3.16a). If $h\in I_\eta$, then, from (3.3), (3.4) and (3.16a), it follows that $\Phi_{\eta,h}\in\Gamma(\Sigma, \mathcal{O}(1))$. Thus, $\Phi_{\eta,h}$ is a constant $c_{\eta,h}$ [22]. Using (3.3), (3.4) and (3.16a) once more, one finds that, when $h>-j_\eta$, $\Phi_{\eta,h-1a}=k_{ab}(\Phi_{\eta,h-1b}-C^{+1}_{j_\eta,h-1}c_{\eta,h}\partial_b \ln k_{ab})$. If $c_{\eta,h}$ were non zero, $-(C^{+1}_{j_\eta,h-1}c_{\eta,h})^{-1}\Phi_{\eta,h-1}$ would be a holomorphic connection of the canonical line bundle k. These are known not to exist [22]. Hence, $c_{\eta,h}=0$. □

Definition 3.3. For any $w\in\mathbb{Z}/2$ with $w\geqslant 0$, let $\{v_i^{(w)}\,|\,i=1, ..., d_w\}$ be a basis of $\Gamma(\Sigma, \mathcal{O}(k^{\otimes w}))$. Further, let R be a holomorphic projective connection. For any $\eta\in\Pi$ and $\mu\in I_\eta$ with either $\mu<h$ or $\mu=h=-j_\eta$ and any $i=1, ..., d_{h-\mu}$, set

$$\Upsilon^{(h)}_{\eta,\mu,i}(R)=F_{h,\eta,\mu}(v_i^{(h-\mu)}\,|R)\ . \tag{3.17}$$

Theorem 3.6. *For any holomorphic projective connection R, the set $\{\Upsilon^{(h)}_{\eta,\mu,i}(R)\,|\,\eta\in\Pi, \mu\in I_\eta$ with either $\mu<h$ or $\mu=h=-j_\eta$, $i=1, ..., d_{h-\mu}\}$ is a basis of $\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes \mathrm{Ad}\,L))$.*

Proof. Since $v_i^{(h-\mu)}\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h-\mu}))$, $\Upsilon^{(h)}_{\eta,\mu,i}(R)\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$ (cf. Thm. 3.3). For $\nu\in\mathbb{N}\cup\{0\}$, let $\Phi_\nu\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$. Let Π_ν be the subset of Π such that, for $\eta\in\Pi_\nu$, $\Phi_{\nu\eta,m}\neq 0$ for some $m\in I_\eta$. For any $\eta\in\Pi_\nu$, let $m_{\nu,\eta}$ be the largest value of $m\in I_\eta$ such that $\Phi_{\nu\eta,m}\neq 0$. By (3.16), either $m_{\nu,\eta}<h$ or $m_{\nu,\eta}=h=-j_\eta$. From (3.3) and (3.4), it follows that $\phi_{\nu,\eta}\equiv \Phi_{\nu\eta,m_{\nu,\eta}}\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h-m_{\nu,\eta}}))$. Applying Thm. 3.3, one can construct an element $F_{h,\eta,m_{\nu,\eta}}(\phi_{\nu,\eta}\,|R)\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$ satisfying (3.5). Set $\Phi_{\nu+1}=\Phi_\nu-\sum_{\eta\in\Pi_\nu}F_{h,\eta,m_{\nu,\eta}}(\phi_{\nu,\eta}\,|R)$. Clearly, $\Phi_{\nu+1}\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$. It is easily checked that $\Pi_{\nu+1}\subseteq\Pi_\nu$ and that, for $\eta\in\Pi_{\nu+1}$, $m_{\nu+1,\eta}<m_{\nu,\eta}$. Using the procedure outlined above, given any element $\Phi\in\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$, one can construct a finite sequence $\Phi_0, \Phi_1, ..., \Phi_{N+1}$ of elements of $\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$ and a finite sequence $\Pi_0, \Pi_1, ..., \Pi_{N+1}$ of subsets of Π such that $\Phi_0=\Phi$, $\Phi_{N+1}=0$ and $\emptyset=\Pi_{N+1}\subseteq\Pi_N\subseteq\cdots\subseteq\Pi_0$. In this way, one reaches the representation

$$\Phi=\sum_{\nu=0}^{N}\sum_{\eta\in\Pi_\nu}F_{h,\eta,m_{\nu,\eta}}(\phi_{\nu,\eta}\,|R)\ . \tag{3.18}$$

From here, it is obvious that the $\Upsilon^{(h)}_{\eta,\mu,i}(R)$ span $\Gamma(\Sigma, \mathcal{O}(k^{\otimes h}\otimes\mathrm{Ad}\,L))$, since $\phi_{\nu,\eta}$ is expressible as a linear combination of the $v_i^{(h-m_{\nu,\eta})}$. Suppose that $\sum_{\eta,\mu,i}c_{\eta,\mu,i}\Upsilon^{(h)}_{\eta,\mu,i}(R)=0$, where $c_{\eta,\mu,i}\in\mathbb{C}$. Then, one also has for each $\eta\in\Pi$, $\sum_{\mu,i}c_{\eta,\mu,i}\Upsilon^{(h)}_{\eta,\mu,i}(R)_{\eta,m}=0$ for $m\in I_\eta$. Let $\mu_{0,\eta}$ be the largest value of μ in the summation range. Now, by (3.5a) and (3.17), $\Upsilon^{(h)}_{\eta,\mu_{0,\eta},i}(R)_{\eta,\mu_{0,\eta}}=v_i^{(h-\mu_{0,\eta})}$. Since the $v_i^{(h)}$ are linearly independent $c_{\eta,\mu_{0\eta},i}=0$ for all i. Let $\mu_{1,\eta}$ be the next to largest value of μ in the summation range. Proceeding as above, one shows that $c_{\eta,\mu_{1,\eta},i}=0$ for all i and so on. □

Theorem 3.7. *One has*

$$\dim \Gamma(\Sigma,\ \mathcal{O}(k^{\otimes h} \otimes \mathrm{Ad}\, L)) = \sum_{\eta \in \Pi, j_\eta \in \{-h, h-1\}} 1 + \sum_{\eta \in \Pi, j_\eta \in \mathbb{Z}+h, j_\eta > \max\{-h, h-1\}} 1$$
$$+ [\dim \Gamma(\Sigma,\ \mathcal{O}(k^{\otimes 1/2})) - \tfrac{1}{4}(l-1)] \sum_{\eta \in \Pi, j_\eta \in \mathbb{Z}+h+1/2, j_\eta > \max\{-h, h-1\}} 1$$
$$+ \left[\sum_{\eta \in \Pi, j_\eta > \max\{-h, h-1\}} (j_\eta + h)^2 + \sum_{\eta \in \Pi, -h < j_\eta \leqslant h-1} (2h-1)(2j_\eta + 1) \right]$$
$$\times (l-1)\,. \tag{3.19}$$

Proof. From Thm. 3.6, it follows readily from here that

$$\dim \Gamma(\Sigma,\ \mathcal{O}(k^{\otimes h} \otimes \mathrm{Ad}\, L)) = \sum_{\eta \in \Pi, j_\eta = -h} \dim \Gamma(\Sigma,\ \mathcal{O}(1))$$
$$+ \sum_{\eta \in \Pi, j_\eta > -h, \mu \in I_\eta, \mu < h} \dim \Gamma(\Sigma,\ \mathcal{O}(k^{\otimes h-\mu}))\,. \tag{3.20}$$

The right hand side of (3.20) can be computed using that $\dim \Gamma(\Sigma,\ \mathcal{O}(k^{\otimes h})) = 1$, l and $(2h-1)(l-1)$, respectively, for $h=0$, $h=1$ and $h \geqslant \frac{3}{2}$ [22]. The calculation is tedious but straightforward. □

3.4. Instability of the G-bundle L

Recall that a holomorphic G-bundle P is unstable if $\dim \Gamma(\Sigma,\ \mathcal{O}(\mathrm{Ad}\, P)) > 0$. This implies in particular the existence of non trivial holomorphic gauge transformations of P, i.e., elements of the group $\Gamma(\Sigma,\ \mathcal{O}^*(\mathrm{Ad}\, P))$ of holomorphic G-valued sections of $\mathrm{Ad}\, P$.

Theorem 3.8. *The G-bundle L is unstable.*

Proof. Indeed, from (3.19) with $h=0$, it follows that $\dim \Gamma(\Sigma,\ \mathcal{O}(\mathrm{Ad}\, L)) > 0$. □

3.5. Flatness and flat structures of L

A holomorphic G-bundle P is said flat if it admits a flat form. Recall that a flat form of a holomorphic G-bundle P is a G-valued constant 1-cocycle T such that $T_{ab} = V_a P_{ab} V_b{}^{-1}$ for some holomorphic G-valued 0-cochain V. It can be shown that P is flat if and only if there is a holomorphic connection of P, i.e., a holomorphic $\mathfrak{g}$-valued 0-cochain C such that $C_a = k_{ab}(\mathrm{Ad}\, P_{ab} C_b + \partial_b P_{ab} P_{ab}{}^{-1})$ [23]. Further, the flat forms of P are in one-to-one correspondence with the holomorphic gauge equivalence classes of holomorphic connections of P, where the action of a holomorphic gauge transformation $\gamma\in\Gamma(\Sigma, \mathcal{O}^*(P))$ on a holomorphic connection C is given by $\gamma C=\mathrm{Ad}\,\gamma C+\partial\gamma\gamma^{-1}$ [23].

Theorem 3.9. *The G-bundle L is flat.*

Proof. This follows from Thm. 3.10 below. □

Definition 3.4. For any holomorphic projective connection R, let $A(R)$ be a $\mathfrak{g}$-valued 0-cochain defined by

$$A(R)_a=\tfrac{1}{2}t_{+1}-R_a t_{-1}\ . \tag{3.21}$$

Theorem 3.10. *For every holomorphic projective connection R, the $\mathfrak{g}$-valued 0-cochain $A(R)$ is a holomorphic connection of L.*

Proof. Indeed, using (2.1) and the relation $k_{ab}{}^{-1}\partial_a{}^2k_{ab}{}^{-1}-\frac{1}{2}(\partial_a k_{ab}{}^{-1})^2=\{z_a, z_b\}$, it is straightforward to verify that

$$A(R)_a=k_{ab}(\mathrm{Ad}\,L_{ab}A(R)_b+\partial_b L_{ab}L_{ab}{}^{-1})\ , \tag{3.22}$$

showing the statement. □

One of the outstanding problems to be tackled is the description of the flat forms of L. I do not have a complete solution of this problem. The answer is expected to depend on the topology of the group G which the method used here, essentially based on Lie algebra theory, cannot probe. In spite of this, a number of results can be shown.

Definition 3.5. An element $\Phi\in\Gamma(\Sigma, \mathcal{O}(\mathrm{Ad}\,L))$ is said to be negative graded if Φ is valued in $\mathfrak{g}_{<0}$. A holomorphic gauge transformation $\gamma\in\Gamma(\Sigma, \mathcal{O}^*(\mathrm{Ad}\,L))$ is said to be negative graded if it is expressible as $\exp\boldsymbol{\Theta}$ for some negative graded element $\boldsymbol{\Theta}\in\Gamma(\Sigma, \mathcal{O}(\mathrm{Ad}\,L))$.

Definition 3.6. A holomorphic connection C of L is said to be reduced if, for some holomorphic projective connection R, $C-A(R)$ is valued in $\ker\mathrm{ad}\,t_{-1}$, where $A(R)$ is the connection (3.21).

From (3.1), one can readily check that these notions are coordinate independent.

Theorem 3.11. *A holomorphic connection C of L is reduced if and only if, for some holomorphic projective connection R, C is of the form*

$$C=A(R)+\sum_{\eta\in\Pi}\omega_\eta t_{\eta,-j_\eta}, \quad \omega_\eta\in\Gamma(\Sigma,\ \mathcal{O}(k^{\otimes j_\eta+1}))\ . \tag{3.23}$$

In that case, C admits the above representation for every holomorphic projective connection R. Hence, the set of reduced holomorphic connections of L can be identified with the affine space $A(R)+\oplus_{\eta\in\Pi}\Gamma(\Sigma,\ \mathcal{O}(k^{\otimes j_\eta+1}))$, *the isomorphism depending on the choice of R.*

Proof. ker ad t_{-1} is spanned by the generators $t_{\eta,-j_\eta}$ with $\eta\in\Pi$. Thus, C is reduced if and only if it is of the form (3.23), for some holomorphic projective connection R. Now, $C-A(R)\in\Gamma(\Sigma,\ \mathcal{O}(k\otimes\mathrm{Ad}\,L))$. Using (3.3) and (3.4), one checks that $\omega_\eta\in\Gamma(\Sigma,\ \mathcal{O}(k^{\otimes j_\eta+1}))$. Finally, from (3.21), it appears that one can change R arbitrarily by redefining ω_0, where $o\in\Pi$ is defined in Section 2. □

Theorem 3.12. *For every holomorphic connection C of L, there is a unique negative graded holomorphic gauge transformation* $\gamma_C\in\Gamma(\Sigma,\ \mathcal{O}^*(\mathrm{Ad}\,L))$ *such that the gauge transformed holomorphic connection* $\hat{C}=\gamma_C C$ *is reduced.*

Proof. Pick a holomorphic projective connection R. For any holomorphic connection C of L, set

$$\Omega(C|R)=C-A(R)\ . \tag{3.24}$$

$\Omega(C|R)\in\Gamma(\Sigma,\ \mathcal{O}(k\otimes\mathrm{Ad}\,L))$. If $\gamma\in\Gamma(\Sigma,\ \mathcal{O}^*(\mathrm{Ad}\,L))$ is a holomorphic gauge transformation, one has

$$\Omega(\gamma C|R)=\mathrm{Ad}\,\gamma\Omega(C|R)+\partial_{A(R)}\gamma\gamma^{-1}, \tag{3.25}$$

where $\partial_{A(R)}=\partial-\mathrm{ad}\,A(R)$ is the covariant derivative associated to the connection $A(R)$ acting on $\Gamma(\Sigma,\ \mathcal{O}(\mathrm{Ad}\,L))$. For $\nu\in\mathbb{N}\cup\{0\}$, let $\Omega_\nu\in\Gamma(\Sigma,\ \mathcal{O}(k\otimes\mathrm{Ad}\,L))$ be of the form

$$\Omega_\nu=\sum_{\eta\in\Pi,j_\eta\leqslant\nu/2-1}\omega_\eta t_{\eta,-j_\eta}+(\text{t.o.d.}<-\nu/2+1)\ , \tag{3.26}$$

where $\omega_\eta\in\Gamma(\Sigma,\ \mathcal{O}(k^{\otimes j_\eta+1}))$ and the abbreviation t.o.d. $<\mu$ denotes terms of t_0-degree less than μ. From (3.3) and (3.4), it follows easily that, for $\eta\in\Pi$ with $j_\eta\in\mathbb{Z}+(\nu-1)/2$ and $j_\eta\geqslant(\nu-1)/2$, $\Omega_{\nu\eta,-(\nu-1)/2}$ belongs to $\Gamma(\Sigma,\ \mathcal{O}(k^{\otimes(\nu+1)/2}))$. Applying Thm. 3.3, one can construct the following negative graded element of $\Gamma(\Sigma,\ \mathcal{O}^*(\mathrm{Ad}\,L))$:

$$\gamma_\nu=\exp\left(\sum_{\eta\in\Pi,j_\eta\in\mathbb{Z}+(\nu-1)/2,j_\eta>(\nu-1)/2}F_{0,\eta,-(\nu+1)/2}(\phi_{\nu,\eta}|R_\nu)\right),$$

$$\phi_{\nu,\eta}=\frac{2}{C^{+1}_{j_\eta,-(\nu+1)/2}}\Omega_{\nu\,\eta,-(\nu-1)/2}\ , \tag{3.27}$$

where R_ν is any chosen holomorphic projective connection. Define

$$\Omega_{\nu+1}=\mathrm{Ad}\,\gamma_\nu\Omega_\nu+\partial_{A(R)}\gamma_\nu\gamma_\nu{}^{-1}\,. \tag{3.28}$$

Using the variational formula

$$\delta \mathrm{e}^X\mathrm{e}^{-X}=\frac{\mathrm{e}^{\mathrm{ad}\,X}-1}{\mathrm{ad}\,X}\,\delta X\,,$$

Eqs. (2.2a) and (3.21) and (3.5a), one finds

$$\begin{aligned}\Omega_{\nu+1}=&-\tfrac{1}{2}\Bigg[t_{+1},\sum_{\eta\in\Pi,j_\eta\in\mathbb{Z}+(\nu-1)/2,j_\eta>(\nu-1)/2}\\
&\quad\times F_{0,\eta,-(\nu+1)/2}(\phi_{\nu,\eta}\,|\,R_\nu)_{\eta,-(\nu+1)/2}t_{\eta,-(\nu+1)/2}\Bigg]\\
&+\sum_{\eta\in\Pi,j_\eta\leqslant\nu/2-1}\omega_\eta t_{\eta,-j_\eta}\\
&+\sum_{\eta\in\Pi,j_\eta\in\mathbb{Z}+(\nu-1)/2,j_\eta\geqslant(\nu-1)/2}\Omega_{\nu\,\eta,-(\nu-1)/2}t_{\eta,-(\nu-1)/2}\\
&+(\text{t.o.d.}<-(\nu+1)/2+1)\\
=&\sum_{\eta\in\Pi,j_\eta\leqslant(\nu+1)/2-1}\omega_\eta t_{\eta,-j_\eta}+(\text{t.o.d.}<-(\nu+1)/2+1)\,,\end{aligned}$$

$$\omega_\eta=\Omega_{\nu\,\eta,-(\nu-1)/2}\,,\quad \eta\in\Pi,j_\eta=(\nu-1)/2\,. \tag{3.29}$$

Thus, $\Omega_{\nu+1}$ is of the form (3.26) with ν replaced by $\nu+1$. From (3.16) with $h=1$, every $\Omega\in\Gamma(\Sigma,\mathcal{O}(k\otimes\mathrm{Ad}\,L))$ is of the form (3.26) with $\nu=0$. So, setting $\Omega_0=\Omega$, one constructs a sequence $\gamma_0,\gamma_1,\dots,\gamma_N$ of negative graded holomorphic gauge transformations of $\Gamma(\Sigma,\mathcal{O}^*(\mathrm{Ad}\,L))$ and a sequence $\Omega_1,\Omega_2,\dots,\Omega_{N+1}$ of elements of $\Gamma(\Sigma,\mathcal{O}(k\otimes\mathrm{Ad}\,L))$, where $N=2j_*+1$, where j_* is defined in Section 2. From (3.26), Ω_{N+1} is of the form

$$\Omega_{N+1}=\sum_{\eta\in\Pi}\omega_\eta t_{\eta,-j_\eta}\,, \tag{3.30}$$

where $\omega_\eta\in\Gamma(\Sigma,\mathcal{O}(k^{\otimes j_\eta+1}))$. Now, take $\Omega=\Omega(C|R)$ and follow the procedure outlined above. Set

$$\gamma_C=\gamma_N\gamma_{N-1}\cdots\gamma_0\,. \tag{3.31}$$

Recall that $\mathfrak{g}_{<0}$ is a nilpotent Lie algebra and that, for a nilpotent Lie algebra, the Hausdorff–Campbell formula holds with no restriction. From these facts, it follows that γ_C is a negative graded element of $\Gamma(\Sigma,\mathcal{O}^*(\mathrm{Ad}\,L))$. From (3.25) and (3.28), one has

$$\Omega(\gamma_C C|R)=\Omega_{N+1}\,. \tag{3.32}$$

Hence, by (3.24) and (3.30), $\gamma_C C$ is reduced. This shows the existence of γ_C. Let $\Omega_1, \Omega_2 \in \Gamma(\Sigma, \mathcal{O}(k \otimes \mathrm{Ad}\, L))$ be of the form

$$\Omega_i = \sum_{\eta \in \Pi} \omega_{i\,\eta} t_{\eta, -j_\eta}, \tag{3.33}$$

with $\omega_{i\,\eta} \in \Gamma(\Sigma, \mathcal{O}(k^{\otimes j_\eta + 1}))$ and let γ be a negative graded element of $\Gamma(\Sigma, \mathcal{O}^*(\mathrm{Ad}\, L))$ such that

$$\Omega_2 = \mathrm{Ad}\, \gamma \Omega_1 + \partial_{A(R)} \gamma \gamma^{-1}. \tag{3.34}$$

γ can be written in the form

$$\gamma = \exp \boldsymbol{\Theta}, \tag{3.35}$$

where $\boldsymbol{\Theta}$ is a negative graded element of $\Gamma(\Sigma, \mathcal{O}(\mathrm{Ad}\, L))$. Combining (3.33) and (3.34) and using the variational formula

$$\delta \mathrm{e}^X \mathrm{e}^{-X} = \frac{\mathrm{e}^{\mathrm{ad}\, X} - 1}{\mathrm{ad}\, X} \delta X,$$

one finds

$$\mathrm{ad}\, t_{-1} \left[\frac{\exp \mathrm{ad}\, \boldsymbol{\Theta} - 1}{\mathrm{ad}\, \boldsymbol{\Theta}} (\partial_{A(R)} \boldsymbol{\Theta} - [\Omega_1, \boldsymbol{\Theta}]) \right] = 0. \tag{3.36}$$

For any $m \in \mathbb{Z}/2$, let π_m be the projector on $\mathfrak{g}_m$ along $\mathfrak{g}_{\neq m}$. Since $\boldsymbol{\Theta}$ is negative graded, $\pi_m \boldsymbol{\Theta} = 0$ for any $m \in \mathbb{Z}/2$ with $m \geqslant 0$. Suppose that $\pi_m \boldsymbol{\Theta} = 0$ for all $m \in \mathbb{Z}/2$ such that $m > n$ where $n \in \mathbb{Z}/2$ with $n < 0$. By grading reasons, recalling (3.21), (3.36) yields

$$0 = -\tfrac{1}{2} [t_{-1}, [t_{+1}, \pi_n \boldsymbol{\Theta}]] + (\text{t.o.d.} < n). \tag{3.37}$$

Using that $\mathrm{ad}\, t_{+1} \mathfrak{g} \cap \ker \mathrm{ad}\, t_{-1} = \{0\}$ and that $\mathfrak{g}_{<0} \cap \ker \mathrm{ad}\, t_{+1} = \{0\}$, one concludes that $\pi_n \boldsymbol{\Theta} = 0$. Proceeding by induction, one shows that $\pi_m \boldsymbol{\Theta} = 0$ for every m. Thus, $\boldsymbol{\Theta} = 0$ and $\gamma = 1$. Now, let γ_1 and γ_2 be two negative graded elements of $\Gamma(\Sigma, \mathcal{O}^*(\mathrm{Ad}\, L))$ such that $\gamma_1 C$ and $\gamma_2 C$ are both reduced. Setting $\Omega_i = \Omega(\gamma_i C | R)$ and $\gamma = \gamma_2 \gamma_1^{-1}$ above, (3.33) and (3.34) hold. So, $\gamma_1 = \gamma_2$. This shows the uniqueness of γ_C. □

Theorem 3.13. *Let $\gamma \in \Gamma(\Sigma, \mathcal{O}^*(\mathrm{Ad}\, L))$ be of the form $\gamma = \exp \boldsymbol{\Theta}$ for some $\boldsymbol{\Theta} \in \Gamma(\Sigma, \mathcal{O}(\mathrm{Ad}\, L))$. Then, γ maps the space of reduced holomorphic connections of L into itself if and only if $\boldsymbol{\Theta} = c$ for some constant element $c \in \mathfrak{c}_s$.*

Proof. To begin with, one notes that, for any $c \in \mathfrak{c}_s$, the $\mathfrak{g}$-valued 0-cochain $\boldsymbol{\Theta}$ defined by $\boldsymbol{\Theta}_a = c$ belongs to $\Gamma(\Sigma, \mathcal{O}(\mathrm{Ad}\, L))$, as follows easily from (3.1). Conversely, from (3.3) and (3.4) and the fact that the only holomorphic functions on Σ are the constants [22], one easily shows that, if $\boldsymbol{\Theta} \in \Gamma(\Sigma, \mathcal{O}(\mathrm{Ad}\, L))$ is valued in $\mathfrak{c}_S$, then $\boldsymbol{\Theta}=c$ for some constant element $c\in\mathfrak{c}_S$. For $c\in\mathfrak{c}_S$, $\operatorname{ad} c \ker \operatorname{ad} t_{-1} \subseteq \ker \operatorname{ad} t_{-1}$. So, if $\boldsymbol{\Theta}=c$ for some constant element $c\in\mathfrak{c}_S$, γ maps the space of reduced holomorphic connections of L into itself. This shows sufficiency. Let Ω_1, $\Omega_2\in\Gamma(\Sigma, \mathscr{O}(k\otimes \operatorname{Ad} L))$ be of the form (3.33) and suppose that (3.34) holds for some holomorphic projective connection R. Then, (3.36) holds as well. Let π_m be defined as below eq. (3.36). By (3.16), $\pi_m\boldsymbol{\Theta}=\delta_{m,0}c$ for $m\in\mathbb{Z}/2$ with $m\geqslant 0$, for some constant element $c\in\mathfrak{c}_S$. If $\pi_m\boldsymbol{\Theta}=\delta_{m,0}c$ for $m\in\mathbb{Z}/2$ such that $m>n$ where $n\in\mathbb{Z}/2$ with $n<0$, then, from (3.21) and (3.36), one gets

$$0=\left[t_{-1}, \frac{\exp \operatorname{ad} c-1}{\operatorname{ad} c}\left(\partial c-[\Omega_1, c]-\tfrac{1}{2}[t_{+1}, \pi_n\boldsymbol{\Theta}]\right)+(\text{t.o.d.}<n+1)\right]$$

$$=-\frac{1}{2}\left[t_{-1}, \frac{\exp \operatorname{ad} c-1}{\operatorname{ad} c}[t_{+1}, \pi_n\boldsymbol{\Theta}]\right]+(\text{t.o.d.}<n)\,. \tag{3.38}$$

Here, I have used that c is constant and that $(\exp \operatorname{ad} c-1)\Omega_1$ is valued in $\ker \operatorname{ad} t_{-1}$. The latter property follows from the fact that Ω_1 is valued in $\ker \operatorname{ad} t_{-1}$ and the already mentioned invariance of $\ker \operatorname{ad} t_{-1}$ under $\operatorname{ad} c$. Reasoning as done below eq. (3.37), one can show that this relation entails that $\pi_n\boldsymbol{\Theta}=0$. Proceeding by induction, one concludes that $\pi_m\boldsymbol{\Theta}=\delta_{m,0}c$ for any $m\in\mathbb{Z}/2$. Thus, $\boldsymbol{\Theta}=c$. If C_1 and C_2 are two reduced holomorphic connections of L such that $\gamma C_1=C_2$, then $\Omega_i=\Omega(C_i|R)$ fulfill the above assumptions. So $\boldsymbol{\Theta}=c$. This shows necessity. □

4. The Drinfeld–Sokolov–Krichever–Novikov spaces and their properties

In the first part of this section, I shall review briefly the main properties of the Krichever–Novikov (KN) spaces, which play an important role in the geometrical framework expounded below. In the second part, I shall introduce the Drinfeld–Sokolov–Krichever–Novikov (DSKN) spaces, describe their standard bases and study their symmetries.

Remark. Throughout this section, the following assumptions are made. Σ is a compact connected Riemann surface without boundary of genus $l\geqslant 2$. $k^{\otimes 1/2}$ is a fixed theta characteristic. h is a fixed element of $\mathbb{Z}/2$. G is a simple complex Lie group. S is an $\mathrm{SL}(2,\mathbb{C})$ subgroup of G.

4.1. The standard KN theory

The basic ingredients of KN theory are the following:

(i) a finite subset Δ of Σ such that $|\Delta|\geqslant 2$ divided into two disjoint subsets Δ_{in} and Δ_{out} such that $|\Delta_{\text{in}}|\geqslant 1$ and $|\Delta_{\text{out}}|\geqslant 1$;

(ii) an element of ρ of $\Gamma(\Sigma, \mathscr{M}(k))$ holomorphic on $\Sigma\backslash\Delta$ with a simple pole of positive (negative) residue at each point of Δ_{in} (Δ_{out}) and imaginary periods. To avoid complication with the Riemann–Roch theory, the points of Δ will be assumed to be in general position [7,24].

Chosing a base point $P_0 \in \Sigma \setminus \Delta$, set

$$t(P) = \mathrm{Re} \int_{P_0}^{P} \rho\,. \tag{4.1}$$

t is a single-valued harmonic function on $\Sigma \setminus \Delta$ with the property that $t(P) \to -\infty$ $(+\infty)$ when P approaches a point of Δ_{in} (Δ_{out}) [7,24]. So, t defines a notion of euclidean time on Σ. For any $\tau \in \mathbb{R}$, the subspace of Σ of time τ is

$$C_\tau = \{P \mid P \in \Sigma \setminus \Delta,\ t(P) = \tau\}\,. \tag{4.2}$$

C_τ is a disjoint union of simple loops in $\Sigma \setminus \Delta$ for all but finitely many critical values of τ, whose number is bounded by $2l - 2 + |\Delta|$. The critical values correspond to processes of topological reconstruction in which either one loop splits into two or more, or two or more loops merge into one. The points of Σ where the reconstruction occurs are precisely the zeros of ρ and the number of loops involved equals the order of the zero plus 2. For any two τ_1, $\tau_2 \in \mathbb{R}$, C_{τ_1} is homologous to C_{τ_2} in $\Sigma \setminus \Delta$. Hence, for any $\omega \in \Gamma(\Sigma, \mathcal{M}(k))$ with poles contained in Δ, $\oint_{C_\tau} \omega$ is τ-independent.

The KN space $\mathsf{KN}_h(\Delta)$ of weight h is the set of the elements of $\Gamma(\Sigma, \mathcal{M}(k^{\otimes h}))$ whose poles are contained in Δ [1]. $\mathsf{KN}_h(\Delta)$ is an infinite dimensional complex vector space.

There exists a bilinear pairing of the spaces $\mathsf{KN}_h(\Delta)$ and $\mathsf{KN}_{1-h}(\Delta)$ defined by

$$\langle \psi, \phi \rangle = \oint_{C_\tau} \frac{dz}{2\pi i} \phi \psi\,, \tag{4.3}$$

for any $\phi \in \mathsf{KN}_h(\Delta)$ and $\psi \in \mathsf{KN}_{1-h}(\Delta)$. Note that the integration is well-defined and independent of $\tau \in \mathbb{R}$.

The space $\mathsf{KN}_h(\Delta)$ possesses a standard basis, the generalized KN basis. To describe this, set $r = |\Delta_{\text{in}}|$, $s = |\Delta_{\text{out}}|$ and $p_h = p_{1-h} = h - [h] \bmod \mathbb{Z}$. The basis is of the form $\{v^{(h)}_{i,N} \mid i = 1, \ldots, r,\ N \in \mathbb{Z} + p_h\}$. The basis elements $v^{(h)}_{i,N}$ are characterized up to normalization by their zero order at the points Δ. Let $\Delta_{\text{in}} = \{P_j \mid 1 \leqslant j \leqslant r\}$ and $\Delta_{\text{out}} = \{P_j \mid r+1 \leqslant j \leqslant r+s\}$. Then,

$$\mathrm{ord}\, v^{(h)}_{i,N}(P_j) = a_j(N + 1 - h) - \delta_{j,i} + (2h-1)(l-1)\delta_{j,r+s} + b_{j,N}\,, \tag{4.4a}$$

where

[1] In a different definition, also essential singularities at the points of Δ are allowed.

$$a_j=\begin{cases}1\,, & 1\leqslant j\leqslant r\,,\\ -1\,, & r+1\leqslant j\leqslant r+\min(r,s)-1\,,\\ -(|r-s|+1)^{\mathrm{sgn}(r-s)}\,, & r+\min(r,s)\leqslant j\leqslant r+s\,,\end{cases}\tag{4.4b}$$

and the $b_{j,N}$ are rational numbers such that

$$\sum_{j=1}^{r+s} b_{j,N}=0\,,\quad |b_{j,N}|<1\,,\quad b_{j,N}=0\,,\quad 1\leqslant j\leqslant r+\min(r,s)-1\,,\tag{4.4c}$$

depending on j and N. These statements about $b_{j,N}$ must be amended for finitely many values of N when h takes the exceptional values 0, $\frac{1}{2}$ for an odd theta characteristic and 1. See ref. [24] for a detailed treatment of this matter and refs. [25–28] for related approaches to the subject.

The relative normalization of the elements of the KN bases of $\mathsf{KN}_h(\Delta)$ and $\mathsf{KN}_{1-h}(\Delta)$ can be chosen so that

$$\langle v^{(1-h)}_{i,M}, v^{(h)}_{j,N}\rangle=\delta_{i,j}\delta_{M,-N}\,,\quad i,j=1,\dots,r\,,\quad M,N\in\mathbb{Z}+p_h\,.\tag{4.5}$$

The Laurent theorem generalizes and one gets the expansion

$$\phi=\sum_{i=1}^{r}\sum_{N\in\mathbb{Z}+p_h}\phi_{i,N}v^{(h)}_{i,-N}\,,\quad \phi_{i,N}=\langle v^{(1-h)}_{i,N},\phi\rangle\,,\tag{4.6}$$

the series containing only a finite number of non vanishing terms [23]. Eqs. (4.5) and (4.6) imply further that the pairing (4.3) is non singular and that the spaces $\mathsf{KN}_h(\Delta)$ and $\mathsf{KN}_{1-h}(\Delta)$ are reciprocally dual.

The basic symmetry group of the KN theory is the conformal group $\mathrm{Conf}_0(\Delta)$, i.e., the group of holomorphic diffeomorphisms f of $\Sigma\setminus\Delta$ onto itself with holomorphic inverse having finite order singularities at the points of Δ and homotopic to id_Σ. Its Lie algebra is $\mathrm{Lie\,Conf}_0(\Delta)=\mathsf{KN}_{-1}(\Delta)$. The Lie brackets are given by

$$[u,v]=u\partial v-v\partial u\,,\tag{4.7}$$

for any two u, $v\in\mathrm{Lie\,Conf}_0(\Delta)$.

$\mathrm{Conf}_0(\Delta)$ acts on the KN spaces $\mathsf{KN}_h(\Delta)$. For any $f\in\mathrm{Conf}_0(\Delta)$ and $\phi\in\mathsf{KN}_h(\Delta)$, the action is defined by

$$f^*\phi_a=k^{\otimes h}(f)_{ab}\phi_b\circ f\,.\tag{4.8}$$

Here,

$$k^{\otimes 1/2}(f)_{ab}=(\partial_a f_b)^{1/2}\,,\tag{4.9}$$

where the branch of the square root is chosen so that

$$k^{\otimes 1/2}(\mathrm{id}_\Sigma)_{ab}=k^{\otimes 1/2}{}_{ab}\,,\tag{4.10a}$$

$$k^{\otimes 1/2}(g\circ f)_{ac}=k^{\otimes 1/2}(f)_{ab}k^{\otimes 1/2}(g)_{bc}\circ f\,,\tag{4.10b}$$

for f, $g\in\mathrm{Conf}_0(\Delta)$. $k^{\otimes h}(f)=k^{\otimes 1/2}(f)^{\otimes 2h}$, by definition. At the infinitesimal level, (4.8) reduces into

$$\theta_u \phi = u\partial\phi + h(\partial u)\phi \tag{4.11}$$

for any $u \in \mathrm{Lie\,Conf}_0(\Delta)$. The KN pairing (4.3) is invariant under $\mathrm{Conf}_0(\Delta)$:

$$\langle f^*\psi, f^*\phi\rangle = \langle \psi, \phi\rangle\ , \tag{4.12}$$

$$\langle \theta_u\psi, \phi\rangle + \langle \psi, \theta_u\phi\rangle = 0\ , \tag{4.13}$$

for any $f \in \mathrm{Conf}_0(\Delta)$, $u \in \mathrm{Lie\,Conf}_0(\Delta)$ and any $\phi \in \mathsf{KN}_h(\Delta)$ and $\psi \in \mathsf{KN}_{1-h}(\Delta)$.

4.2. The DSKN spaces

Definition 4.1. The DSKN space $\mathsf{DS}_h(\Delta)$ of weight h is the set of the elements of $\Gamma(\Sigma, \mathcal{M}(k^{\otimes h} \otimes \mathrm{Ad}\, L))$ whose poles are contained in Δ.

$\mathsf{DS}_h(\Delta)$ is an infinite dimension complex vector space.

Definition 4.2. For any $\Phi \in \mathsf{DS}_h(\Delta)$ and $\Psi \in \mathsf{DS}_{1-h}(\Delta)$, set

$$\langle \Psi, \Phi\rangle = \oint_{C_\tau} \frac{\mathrm{d}z}{2\pi \mathrm{i}}\,(\Phi, \Psi)\ . \tag{4.14}$$

Note that the integration is well-defined and independent of $\tau \in \mathbb{R}$. Eq. (4.14) defines a bilinear pairing of the spaces $\mathsf{DS}_h(\Delta)$ and $\mathsf{DS}_{1-h}(\Delta)$, called the DSKN pairing. It is the appropriate generalization of the customary KN pairing in the present context.

4.3. The DSKN bases

The space $\mathsf{DS}_h(\Delta)$ admits standard bases. To construct them, one needs the following result.

Recall that a meromorphic connection ϖ of k on Σ is a meromorphic 0-cochain $\{\varpi_a\}$ on Σ such that $\varpi_a = k_{ab}(\varpi_b + \partial_b \ln k_{ab})$.

Definition 4.3. Let ϖ be a meromorphic connection of k on Σ whose poles are contained in Δ. Let further $\eta \in \Pi$, $\mu \in I_\eta$ and $\phi \in \mathsf{KN}_{h-\mu}(\Delta)$. One sets

$$Q_{h,\eta,\mu}(\phi|\varpi) = \exp(-\varpi\,\mathrm{ad}\, t_{-1})\phi t_{\eta,\mu}\ . \tag{4.15}$$

Theorem 4.1. *For any meromorphic connection ϖ of k on Σ whose poles are contained in Δ and any $\eta \in \Pi$, $\mu \in I_\eta$ and $\phi \in \mathsf{KN}_{h-\mu}(\Delta)$, $Q_{h,\eta,\mu}(\phi|\varpi) \in \mathsf{DS}_h(\Delta)$. Further, $Q_{h,\eta,\mu}(\phi|\varpi)$ depends linearly on ϕ.*

Proof. Indeed, using (2.1), one has

$$\exp(-\varpi_a \operatorname{ad} t_{-1})t_{\eta,\mu} = k^{\otimes\mu}{}_{ab} \operatorname{Ad} L_{ab}(\exp(-\varpi_b \operatorname{ad} t_{-1})t_{\eta,\mu}) . \tag{4.16}$$

From here, the statement is obvious. □

Definition 4.4. Let ϖ be a meromorphic connection of k on Σ whose poles are contained in Δ. For any $\eta \in \Pi$, $\mu \in I_\eta$, $i = 1, \ldots, r$ and $N \in \mathbb{Z} + p_h$, set

$$\Upsilon^{(h)}_{\eta,\mu,i,N}(\varpi) = Q_{h,\eta,\mu}(v^{(h-\mu)}_{i,N} \mid \varpi) . \tag{4.17}$$

Theorem 4.2. *Let ϖ be a meromorphic connection of k on Σ whose poles are contained in Δ. Then, the set $\{\Upsilon^{(h)}_{\eta,\mu,i,N}(\varpi) \mid \eta \in \Pi, \mu \in I_\eta, i = 1, \ldots, r, N \in \mathbb{Z} + p_h\}$ is a basis of $\mathsf{DS}_h(\Delta)$. The basis elements satisfy the relation*

$$\langle \Upsilon^{(1-h)}_{\eta,\mu,i,M}(\varpi), \Upsilon^{(h)}_{\zeta,\nu,j,N}(\varpi) \rangle = N_\eta \delta_{\eta,\zeta}(-1)^{j_\eta - \mu} \delta_{\mu,-\nu} \delta_{i,j} \delta_{M,-N} ,$$
$$\eta, \zeta \in \Pi , \quad \mu \in I_\eta , \nu \in I_\zeta , \quad i, j = 1, \ldots, r , \quad M, N \in \mathbb{Z} + p_h . \tag{4.18}$$

Finally, for any $\Phi \in \mathsf{DS}_h(\Delta)$, one has

$$\Phi = \sum_{\eta \in \Pi, \mu \in I_\eta} \sum_{i=1}^{r} \sum_{N \in \mathbb{Z} + p_h} \Phi_{\eta,\mu,i,N}(\varpi) \Upsilon^{(h)}_{\eta,-\mu,i,-N}(\varpi) ,$$
$$\Phi_{\eta,\mu,i,N}(\varpi) = \langle \Upsilon^{(1-h)}_{\eta,\mu,i,N}(\varpi), \Phi \rangle / N_\eta (-1)^{j_\eta - \mu} , \tag{4.19}$$

the series containing only a finite number of non vanishing terms.

Proof. Let $\Phi \in \mathsf{DS}_h(\Delta)$. Following a procedure totally analogous to that leading to (3.18), one shows that

$$\Phi = \sum_{\nu=0}^{K} \sum_{\eta \in \Pi_\nu} Q_{h,\eta,m_{\nu,\eta}}(\phi_{\nu,\eta} \mid \varpi) , \tag{4.20}$$

where K is some non negative integer, Π_ν is a subset of Π and $\phi_{\nu,\eta} \in \mathsf{KN}_{h-m_{\nu,\eta}}(\Delta)$ for each ν and η. Each $\phi_{\nu,\eta}$ is given by a series of the form (4.6). Hence, the $\Upsilon^{(h)}_{\eta,\mu,i,N}(\varpi)$ span $\mathsf{DS}_h(\Delta)$. The linear independence of the $\Upsilon^{(h)}_{\eta,\mu,i,N}(\varpi)$ is equivalent to that of the fields $v^{(h)}_{i,N} t_{\eta,\mu}$, which is obvious. Eq. (4.18) is a straightforward consequence of (4.14), (4.5) and (2.10). Eq. (4.19) follows from the representation (4.18) and from (4.6). □

Theorem 4.3. *The pairing (4.14) is non singular. Therefore, the spaces $\mathsf{DS}_h(\Delta)$ and $\mathsf{DS}_{1-h}(\Delta)$ are reciprocally dual.*

Proof. This follows directly from (4.19) and (4.20). □

4.4. The symmetries of the DSKN spaces

There exists a natural extension of the action of $\mathrm{Conf}_0(\Delta)$ to the DSKN spaces. This leads to the following result.

Theorem 4.4. $\mathrm{Conf}_0(\Delta)$ *acts on the space* $\mathsf{DS}_h(\Delta)$ *by setting*

$$f^*\Phi_a = k^{\otimes h}(f)_{ab}\,\mathrm{Ad}\,L(f)_{ab}\Phi_b \circ f \tag{4.21}$$

for arbitrary $f \in \mathrm{Conf}_0(\Delta)$ *and* $\Phi \in \mathsf{DS}_h(\Delta)$, *where*

$$L(f)_{ab} = \exp(-2\ln k^{\otimes 1/2}(f)_{ab} t_0)\exp(\partial_a k(f)_{ab}{}^{-1} t_{-1}) . \tag{4.22}$$

At the infinitesimal level, (4.21) can be written as

$$\theta_u \Phi = u\partial_{A(R)}\Phi + h(\partial u)\Phi + [\dot{L}(u|R), \Phi] , \tag{4.23}$$

for any $u \in \mathrm{Lie\,Conf}_0(\Delta)$, *where* R *is a holomorphic projective connection and*

$$\dot{L}(u|R) = [\tfrac{1}{2}t_{+1} - \partial t_0 - (\partial^2 + R)t_{-1}]u , \tag{4.24}$$

$\partial_{A(R)} = \partial - \mathrm{ad}\,A(R)$ *being the covariant derivative of the connection* $A(R)$ *defined in* (3.21). $\dot{L}(u|R) \in \mathsf{DS}_0(\Delta)$ *and* $\dot{L}(u|R)$ *satisfies the equation*

$$\partial_{A(R)}\dot{L}(u|R) = -D_1(R)ut_{-1} , \quad D_1(R) = \partial^3 + 2R\partial + (\partial R) , \tag{4.25}$$

where $D_1(R)$ *is a Bol operator [21].*

Proof. From (4.10a), it is easily verified that

$$L(\mathrm{id}_\Sigma)_{ab} = L_{ab} . \tag{4.26a}$$

From (2.1) and (4.10b), one verifies further that

$$L(g \circ f)_{ac} = L(f)_{ab} L(g)_{bc} \circ f , \tag{4.26b}$$

for any $f, g \in \mathrm{Conf}_0(\Delta)$. Using (4.10) and (4.26) in combination, it is straightforward to verify that the right hand side of (4.21) belongs to $\mathsf{DS}_h(\Delta)$. The remaining statements are straightforwardly verified. □

Note that $\theta_u \Phi$ is independent of R. R is introduced only in order that the various contributions appearing in its expression have nice covariance properties.

Theorem 4.5. *The pairing (4.14) is invariant under* $\mathrm{Conf}_0(\Delta)$. *In fact, one has*

$$\langle f^*\Psi, f^*\Phi \rangle = \langle \Psi, \Phi \rangle , \tag{4.27}$$

$$\langle \theta_u \Psi, \Phi \rangle + \langle \Psi, \theta_u \Phi \rangle = 0 , \tag{4.28}$$

for any $f \in \mathrm{Conf}_0(\Delta)$, $u \in \mathrm{Lie\,Conf}_0(\Delta)$ *and any* $\Phi \in \mathsf{DS}_h(\Delta)$ *and* $\psi \in \mathsf{DS}_{1-h}(\Delta)$.

Proof. The verification is straightforward. □

Denote by $\mathrm{Gau}_0(\Delta)$ the group of gauge transformations $\gamma \in \Gamma(\Sigma \setminus \Delta, \mathscr{M}^*(\mathrm{Ad}\, L))$ with finite order singularities at the points of Δ and homotopic to the identity. Its Lie algebra $\mathrm{Lie\,Gau}_0(\Delta)$ is $\mathsf{DS}_0(\Delta)$ with Lie brackets

$$[\Xi, \Lambda] = [e(\Xi), e(\Lambda)], \tag{4.29}$$

for any two $\Xi, \Lambda \in \mathrm{Lie\,Gau}_0(\Delta)$, where in the right hand side e is the evaluation map at a given point of $\Sigma \setminus \Delta$ and the Lie brackets are those of $\mathfrak{g}$.

$\mathrm{Gau}_0(\Delta)$ acts on the DSKN spaces by means of the adjoint representation, as summarized by the following theorem.

Theorem 4.6. $\mathrm{Gau}_0(\Delta)$ *acts on the space* $\mathsf{DS}_h(\Delta)$ *by setting*

$$\gamma\Phi = \mathrm{Ad}\,\gamma\Phi \tag{4.30}$$

for arbitrary $\gamma \in \mathrm{Gau}_0(\Delta)$ *and* $\Phi \in \mathsf{DS}_h(\Delta)$. *At the infinitesimal level, (4.30) becomes*

$$\delta_\Xi \Phi = [\Xi, \Phi] \tag{4.31}$$

for any $\Xi \in \mathrm{Lie\,Gau}_0$.

Proof. The verification is straightforward. □

Theorem 4.7. *The pairing* (4.14) *is invariant under* $\mathrm{Gau}_0(\Delta)$. *In fact, one has*

$$\langle \gamma\Psi, \gamma\Phi \rangle = \langle \Psi, \Phi \rangle, \tag{4.32}$$

$$\langle \delta_\Xi \Psi, \Phi \rangle + \langle \Psi, \delta_\Xi \Phi \rangle = 0 \tag{4.33}$$

for any $\gamma \in Gau_0(\Delta)$, $\Xi \in \mathrm{Lie\,Gau}_0(\Delta)$ *and any* $\Phi \in \mathsf{DS}_h(\Delta)$ *and* $\Psi \in \mathsf{DS}_{1-h}(\Delta)$.

Proof. This verification is also straightforward. □

Note that the total symmetry group is the semidirect product $\mathrm{Conf}_0(\Delta) \ltimes \mathrm{Gau}_0(\Delta)$, where the product is defined by $(f_1, \gamma_1) \circ (f_2, \gamma_2) = (f_1 \circ f_2, \gamma_1 f_1{}^{-1*} \gamma_2)$, for $f_1, f_1 \in \mathrm{Conf}_0(\Delta)$ and $\gamma_1, \gamma_2 \in \mathrm{Gau}_0(\Delta)$. The action of the first and second factors are respectively right and left.

5. The Poisson manifold $(\mathsf{W}, \{\cdot, \cdot\}_\kappa)$

In this section, I shall introduce a Poisson manifold $(\mathsf{W}, \{\cdot, \cdot\}_\kappa)$ which is closely related to the customary Kac–Moody phase space, though its geometry is in some respects quite different. In fact, the construction uses the DS bundle L and the

DSKN spaces in an essential way.

Remark. In this section, Σ is a compact connected Riemann surface without boundary of genus $l \geqslant 2$. G is a simple complex Lie group. S is an $\mathrm{SL}(2, \mathbb{C})$ subgroup of G.

In the application of the KN theory below, Δ consists of just two points P_+ and P_- in general position. Δ_{in} and Δ_{out} contain respectively the point P_- and P_+. ρ is the unique element of $\Gamma(\Sigma, \mathcal{M}(k))$ holomorphic on $\Sigma \setminus \Delta$ with a simple pole of residue $+1$ (-1) at P_+ (P_-) and imaginary periods.

To lighten the notation, the dependence of the functional spaces encountered below on Δ will be omitted.

The relevant space of the construction is

$$\mathsf{W} = \mathsf{DS}_1 , \tag{5.1}$$

that is the DSKN space of weight 1. W is an infinite dimensional complex vector space and, thus, also an infinite dimensional holomorphic manifold. The relevant function space on W is the space $\mathfrak{D}(\mathsf{W})$ of differential polynomials on W. W can be endowed with a Poisson structure depending on the parameter $\kappa \in \mathbb{C} \setminus \{0\}$ and supported on $\mathfrak{D}(\mathsf{W})$. The Poisson structure is completely defined by assigning the Poisson brackets of the linear inhomogeneous functionals on W. The Poisson brackets of general elements of $\mathfrak{D}(\mathsf{W})$ are obtained by enforcing the Leibniz rule. This leads to considering the dual space $\mathsf{W}^\vee$ of W. Under the non singular DSKN pairing (4.14), one has the identification

$$\mathsf{W}^\vee = \mathsf{DS}_0 . \tag{5.2}$$

Therefore, every linear functional on W is of the form

$$\lambda_X(W) = \langle X, W \rangle , \quad W \in \mathsf{W} , \tag{5.3}$$

for some $X \in \mathsf{W}^\vee$. Note that $\mathsf{W}^\vee$ has an obvious structure of Lie algebra.

For any $X, Y \in \mathsf{W}^\vee$ and any $a, b \in \mathbb{C}$, the Poisson brackets of the inhomogeneous linear functionals $\lambda_X + a$ and $\lambda_Y + b$ are given by

$$\{\lambda_X + a, \lambda_Y + b\}_\kappa = \lambda_{[X,Y]} + \kappa \chi(X, Y) , \tag{5.4a}$$

$$\chi(X, Y) = \langle X, \partial_A Y \rangle , \tag{5.4b}$$

where A is the holomorphic connection of L given by (3.21) for some holomorphic projective connection R. R will be fixed once and for all in the following. So, the dependence on R will be understood below, to simplify the notation.

It is straightforward to verify that the Poisson brackets $\{\cdot, \cdot\}_\kappa$ are bilinear, antisymmetric and satisfy the Jacobi identity as they should. In fact, one easily checks that χ is a Lie algebra 1-cocycle of $\mathsf{W}^\vee$. χ depends on the choice of R, but changing the choice alters χ by a trivial 1-cocycle. χ is singular, since $\chi(X, Y) = 0$ identically whenever either X or Y are constant elements of $\mathfrak{c}_5$.

The above Poisson structure provides the proper geometric definition of Kac–Moody phase space in the present context. The level and the Kac–Moody current correspond to $-\kappa$ and $\kappa A+W$, respectively.

Next, one has to consider the symmetries of the Poisson manifold W. These are given by suitable deformations of the conformal and gauge symmetries introduced in Section 4.

Consider Conf_0. For any $f\in\mathrm{Conf}_0$, one has that $\chi(f^*X, f^*Y)=\chi(X,Y)-\kappa\langle[X,Y], A-f^{-1*}A\rangle$, where for $f\in\mathrm{Conf}_0$, $f^*A=\partial L(f)L(f)^{-1}+\partial f\,\mathrm{Ad}\,L(f)A\circ f$ and $X, Y\in \mathrm{W}^\vee$. Because of the non invariance of χ, the action of Conf_0 on W, defined by (4.21), is not Poisson: it does not leave the Poisson brackets invariant. However, there is a deformation of the action enjoying this property. Set

$$(f^*W)_\kappa=f^*W+\kappa(f^*A-A)\,,\quad W\in\mathrm{W}\,. \tag{5.5}$$

The deformation induces an action of Conf_0 on $\mathfrak{D}(\mathrm{W})$. It is sufficient to consider the action on the functionals λ_X+a, $X\in\mathrm{W}^\vee$, $a\in\mathbb{C}$, which is given by

$$\begin{aligned}(f^*(\lambda_X+a))_\kappa(W)&=\lambda_X((f^{-1*}W)_\kappa)+a\\ &=\lambda_{f^*X}(W)+a+\kappa\langle X, f^{-1*}A-A\rangle\,,\quad W\in\mathrm{W}\,.\end{aligned} \tag{5.6}$$

From (5.4) and (5.6), it follows that the action (5.5) is Poisson.

At the infinitesimal level, (5.5) and (5.6) become

$$(\theta_u W)_\kappa=\kappa\partial_A\dot{L}(u)+\theta_u W\,, \tag{5.7}$$

$$(\theta_u(\lambda_X+a))_\kappa(W)=-\lambda_X((\theta_u W)_\kappa)=\lambda_{\theta_u X}(W)+\kappa\langle\dot{L}(u),\partial_A X\rangle\,, \tag{5.8}$$

where $u\in\mathrm{Lie\,Conf}_0$ and $\dot{L}(u)$ is given by (4.24). The action is Hamiltonian. In fact, using (5.4), it is straightforward to verify that

$$(\theta_u(\lambda_X+a))_\kappa=\{T_u,\lambda_X+a\}_\kappa\,, \tag{5.9a}$$

$$T_u(W)=(1/2\kappa)\langle uW, W\rangle+\langle\dot{L}(u), W\rangle\,,\quad W\in\mathrm{W}\,, \tag{5.9b}$$

the Hamiltonian functions T_u being elements of $\mathfrak{D}(\mathrm{W})$. T_u can be written as

$$T_u(W)=\langle u, T(W)\rangle\,,\quad W\in\mathrm{W}\,, \tag{5.10a}$$

$$T(W)=(\tfrac{1}{2}t_{+1}+\partial t_0-(\partial^2+R)t_{-1}, W)+(1/2\kappa)(W,W)\,. \tag{5.10b}$$

Using (3.1) and (2.1) and (2.9), it is straightforward albeit lengthy to check that $T(W)\in\mathrm{KN}_2$. So, the map $W\in\mathrm{W}\to T(W)\in\mathrm{KN}_2$ is the moment map of the Hamiltonian action.

For any $\gamma\in\mathrm{Gau}_0$, one has that $\chi(\gamma X,\gamma Y)=\chi(X,Y)-\langle[X,Y],\gamma^{-1}\partial_A\gamma\rangle$. So, the ordinary action of Gau_0 on W, defined by (4.30), is not Poisson. However, in this case too, there exists a deformation of the action enjoying such property, namely

$$(\gamma W)_\kappa = \gamma W + \kappa \partial_A \gamma \gamma^{-1}, \quad W \in \mathsf{W}. \tag{5.11}$$

The deformation induces an action of Gau_0 on the functionals $\lambda_X + a$, $X \in \mathsf{W}^\vee$, $a \in \mathbb{C}$:

$$\begin{aligned}(\gamma(\lambda_X + a))_\kappa(W) &= \lambda_X((\gamma^{-1}W)_\kappa) + a \\ &= \lambda_{\gamma X}(W) + a + \kappa\langle X, \partial_A \gamma^{-1}\gamma\rangle, \quad W \in \mathsf{W}.\end{aligned} \tag{5.12}$$

By combining (5.4) and (5.12), one verifies that the deformed action thus defined is Poisson. At the infinitesimal level, (5.11) and (5.12) become

$$(\delta_\Xi W)_\kappa = \delta_\Xi W + \kappa \partial_A \Xi, \tag{5.13}$$

$$(\delta_\Xi(\lambda_X + a))_\kappa(W) = -\lambda_X((\delta_\Xi W)_\kappa) = \lambda_{\delta_\Xi X}(W) + \kappa\langle \Xi, \partial_A X\rangle, \tag{5.14}$$

where $\Xi \in \mathrm{Lie}\,\mathrm{Gau}_0$ (cf. Eq. (4.31)). From (5.4) and (5.14), one has

$$(\delta_\Xi(\lambda_X + a))_\kappa = \{J_\Xi, \lambda_X + a\}_\kappa, \tag{5.15a}$$

$$J_\Xi(W) = \langle \Xi, W\rangle, \quad W \in \mathsf{W}. \tag{5.15b}$$

Note that $J_\Xi \in \mathfrak{D}(\mathsf{W})$. From here, it appears that the deformed action of Gau_0 on W is Hamiltonian with respect to the Poisson structure (5.4), the Hamiltonian functions being the J_Ξ. J_Ξ can trivially be written as

$$J_\Xi(W) = \langle \Xi, J(W)\rangle, \quad W \in \mathsf{W}. \tag{5.16a}$$

$$J(W) = W. \tag{5.16b}$$

So, the map $W \in \mathsf{W} \to J(W) \in \mathsf{W}$ can be identified with the moment map of the Hamiltonian action.

By a straightforward calculation, one obtains

$$\{T_u, T_v\}_\kappa = T_{[u,v]} + 12\kappa(t_0, t_0)\sigma(u, v), \quad u, v \in \mathrm{Lie}\,\mathrm{Conf}_0, \tag{5.17a}$$

where

$$\sigma(u, v) = -\tfrac{1}{12}\langle u, D_1 v\rangle, \tag{5.17b}$$

is the KN 1-cocycle and D_1 is given in (4.25). The proof of (5.17) uses (5.4), (5.9b), (2.9) and the following two relations:

$$u\partial_A \dot{L}(v) - v\partial_A \dot{L}(u) + [\dot{L}(u), \dot{L}(v)] = \dot{L}([u, v]), \tag{5.18}$$

$$\chi(\dot{L}(u), \dot{L}(v)) = -(t_0, t_0)\langle u, D_1 v\rangle, \tag{5.19}$$

which are easily verified using (2.1) and (4.24), (4.25). (5.17) is a Poisson bracket Virasoro algebra of central charge $12\kappa(t_0, t_0)$. This is the well-known value of the classical central charge encountered in the theory of classical W-algebras [1,9–13]. The moment map $T(W)$, Eq. (5.10b), is the energy–momentum tensor. In the usual approach [9–13], the central charge originates from an improvement term added to the Sugawara energy–momentum tensor of Kac–Moody theory in order to maintain conformal invariance upon carrying out the Hamiltonian reduction of the Kac–Moody phase space. The first and second contributions in expression (5.9b) of T_v correspond more or less to such terms in the present formulation. Here, however, the improvement term is yielded ab initio by the nature of the DS vector bundle and the action of the conformal group of $\Sigma\backslash\Delta$. The second derivative term appearing in expression (5.10b) of $T(W)$ has a counterpart in the usual approach where it is added ad hoc after the reduction of the phase space [9–13]. Here, it is present from the beginning and it is strictly necessary to ensure the correct transformation properties of $T(W)$ under coordinate changes.

From (5.4) and (5.15b), one gets

$$\{J_\Xi, J_\Lambda\}_\kappa = J_{[\Xi,\Lambda]} + \kappa\chi(\Xi,\Lambda)\ , \quad \Xi, \Lambda \in \mathrm{Lie\,Gau}_0\ , \tag{5.20}$$

(5.20) is a Poisson bracket Kac–Moody algebra of level-κ. The moment map $J(W)$, Eq. (5.16b), plays here the role of the Kac–Moody current.

From (5.4), (5.9b) and (5.15b), one also has

$$\{T_u, J_\Xi\}_\kappa = J_{\theta_u \Xi} + \kappa\chi(\dot{L}(u), \Xi)\ , \quad u \in \mathrm{Lie\,Conf}_0, \Xi \in \mathrm{Lie\,Gau}_0\ . \tag{5.21}$$

Hence, the current $J(W)$ transforms as a primary field under Poisson bracketting, except for the component corresponding to the generator t_{+1} of $\mathfrak{g}$ (see Eqs. (4.25) and (5.4b)). This also is familiar in the theory of classical W-algebras [9–12].

It is interesting to write the Poisson bracket algebra in modes. One uses the KN $\{v_{i,N}^{(h)}\}$ and the DSKN bases $\{\Upsilon^{(h)}_{\eta,\mu,i,N}(\varpi)\}$ introduced in Section 4, where ϖ is a meromorphic connection of k holomorphic on $\Sigma\backslash\Delta$. In this case, r being 1, one can suppress the index i. To simplify the notation, the dependence on ϖ will be understood. Set

$$T_P = T_{v_P^{(-1)}}\ , \tag{5.22}$$

$$J_{\eta,\mu,M} = J_{\Upsilon^{(0)}_{\eta,\mu,M}}\ , \tag{5.23}$$

for $P \in \mathbb{Z}$ and $\eta \in \Pi$, $\mu \in I_\eta$ and $M \in \mathbb{Z} + p_{j_\eta}$. Then, (5.17)–(5.21) yield the following algebra:

$$\{T_P, T_Q\}_\kappa = \sum_{R\in\mathbb{Z}} c_P^{(-1)}{}_Q^{(-1)}{}^R_{(-1)} T_R - \kappa(t_0, t_0)\zeta_P^{(-1)}{}_Q^{(-1)}\ , \tag{5.24}$$

$$\begin{aligned}
&\{J_{\eta,\mu,M}, J_{\zeta,\nu,N}\}_\kappa \\
&\quad = \sum_{\xi\in\Pi} \sum_{L\in\mathbb{Z}+p_{j_\xi}} F_{\eta,\zeta}{}^\xi(j_\eta, \mu; j_\zeta, \nu \,|\, j_\xi, \mu+\nu) f_M^{(-\mu)}{}_N^{(-\nu)}{}^L_{(-\mu-\nu)} J_{\xi,\mu+\nu,L} \\
&\qquad + \kappa N_\eta \delta_{\eta,\bar\zeta}(-1)^{j_\eta-\mu}[-\tfrac{1}{2}C^{+1}_{j_\zeta,\nu}\delta_{\mu+\nu+1,0}\delta_{M+N,0}
\end{aligned}$$

$$+C^{-1}_{j\zeta,\nu}\delta_{\mu+\nu-1,0}q^{(-1+\nu)}_{M}{}^{(-\nu)}_{N}+\delta_{\mu+\nu,0}\chi^{(\nu)}_{M}{}^{(-\nu)}_{N}] , \tag{5.25}$$

$$\{T_P, J_{\eta,\nu,N}\}_\kappa$$
$$= \sum_{L\in\mathbb{Z}+p_{j_\eta}} [c^{(-1)}_{P}{}^{(-\nu)}_{N}{}^{L}_{(-\nu)}J_{\eta,\nu,L} - C^{-1}_{j\eta,\nu}g^{(-1)}_{P}{}^{(-\nu)}_{N}{}^{L}_{(-\nu+1)}J_{\eta,\nu-1,L}]$$
$$+2^{1/2}\kappa(t_0, t_0)\zeta^{(-1)}_{P}{}^{(-1)}_{N}\delta_{\eta,o}\delta_{\nu,1} , \tag{5.26}$$

for $P, Q\in\mathbb{Z}$ and $\eta, \zeta\in\Pi$, $\mu\in I_\eta$, $\nu\in I_\zeta$, $M\in\mathbb{Z}+p_{j_\eta}$ and $N\in\mathbb{Z}+p_{j_\zeta}$. Here,

$$c^{(m)}_{M}{}^{(n)}_{N}{}^{L}_{(1+m+n)} = \langle -mv^{(m)}_M\otimes\partial v^{(n)}_N + nv^{(n)}_N\otimes\partial v^{(m)}_M, v^{(-m-n)}_{-L}\rangle , \tag{5.27a}$$

$$f^{(m)}_{M}{}^{(n)}_{N}{}^{L}_{(m+n)} = \langle v^{(m)}_M\otimes v^{(n)}_N, v^{(1-m-n)}_{-L}\rangle , \tag{5.27b}$$

$$g^{(m)}_{M}{}^{(n)}_{N}{}^{L}_{(m+n+2)} = \langle \partial_\varpi\partial_\varpi v^{(m)}_M\otimes v^{(n)}_N, v^{(-1-m-n)}_{-L}\rangle , \tag{5.27c}$$

for $m, n\in\mathbb{Z}/2$ and $M\in\mathbb{Z}+p_m$, $N\in\mathbb{Z}+p_n$ and $L\in\mathbb{Z}+p_{m+n}$ and

$$\zeta^{(-1)}_{P}{}^{(-1)}_{Q} = \langle v^{(-1)}_M, D_1 v^{(-1)}_Q\rangle , \tag{5.28a}$$

$$\chi^{(-m)}_{M}{}^{(m)}_{N} = \langle v^{(-m)}_M, \partial_\varpi v^{(m)}_N\rangle , \tag{5.28b}$$

$$q^{(-1-m)}_{M}{}^{(m)}_{N} = \langle (R-R_\varpi)\otimes v^{(-1-m)}_M, v^{(m)}_N\rangle , \tag{5.28c}$$

for $P, Q\in\mathbb{Z}$ and $m\in\mathbb{Z}/2$ and $M, N\in\mathbb{Z}+p_m$. $o\in\Pi$, $F_{\eta,\zeta}{}^{\xi}$ and N_η are defined in Section 2 (cf. Eqs. (2.3) and (2.10)). D_1 is defined in (4.24). ∂_ϖ is the covariant derivative of the connection ϖ: $\partial_\varpi\phi=(\partial-m\varpi)\phi$ for $\phi\in\mathsf{KN}_m$. R_ϖ is the meromorphic projective connection associated to ϖ: $R_\varpi=\partial\varpi-(1/2)\varpi^2$. Assume now that the poles of the meromorphic connection ϖ are simple. Then, the structure constants $f^{(m)}_{M}{}^{(n)}_{N}{}^{L}_{(m+n)}$, $c^{(m)}_{M}{}^{(n)}_{N}{}^{L}_{(1+m+n)}$ and $g^{(m)}_{M}{}^{(n)}_{N}{}^{L}_{(m+n+2)}$ vanish unless $0\leqslant L-M-N\leqslant l$, $3l$, $5l$, respectively, whenever the values of the weights, written within parentheses, do not take the exceptional values 0, $\frac{1}{2}$ for an odd theta characteristic and 1, while they are non zero only for finitely many values of $L-M-N$ for the exceptional values of the weights. Similarly, $\chi^{(-m)}_{M}{}^{(m)}_{N}$, $q^{(-1-m)}_{M}{}^{(m)}_{N}$ and $\zeta^{(-1)}_{M}{}^{(-1)}_{N}$ vanish unless $0\leqslant -M-N\leqslant 2l$, $4l$, $6l$, respectively, for non exceptional values of the weights, and are non zero only for finitely many values of $M+N$ for the exceptional values of the weights. The calculation yielding the above formula uses (2.1), (2.2), (2.3) and (2.10) and (3.21) and is straightforward.

6. The reduction of the Poisson manifold $(\mathsf{W}, \{\cdot, \cdot\}_\kappa)$

To obtain the classical W-algebras in the above framework, one has to impose a suitable set of first class constraints on the Poisson manifold $(\mathsf{W}, \{\cdot, \cdot\}_\kappa)$ and then fix the gauge to reduce it. This is the subject of this section.

Remark. In this section, Σ, G, S, Δ and ρ are defined as in Section 5.

The constraints imposed are linear. Their general form is

$$J_{\Xi} \approx 0\,, \qquad \Xi \in \mathsf{X}\,, \tag{6.1}$$

where X is some subset of $\mathrm{Lie\,Gau}_0$ and $\approx$ denotes weak equality. Such constraints are essentially of the same form as those used in Ref. [10] once one recalls that in the present formulation the counterpart of the Kac–Moody current is $\kappa A + J(W)$. To implement the reduction of $(\mathsf{W}, \{\cdot,\cdot\}_\kappa)$, one demands that the constraints are first class. From (5.20), this yields the condition

$$[\Xi, \Lambda] \in \mathsf{X} \quad \text{and} \quad \chi(\Xi, \Lambda) = 0\,, \quad \Xi, \Lambda \in \mathsf{X}\,. \tag{6.2}$$

One also requires that the constraint manifold is invariant under the action of Conf_0. From (5.21), this yields the condition

$$\theta_u \Xi \in \mathsf{X} \quad \text{and} \quad \chi(\dot{L}(u), \Xi) = 0\,, \quad u \in \mathrm{Lie\,Conf}_0, \Xi \in \mathsf{X}\,. \tag{6.3}$$

A maximal subspace of X of $\mathrm{Lie\,Gau}_0$ satisfying (6.2) and (6.3) is obtained as follows. The treatment given here follows very closely that of Ref. [10]. Consider the 2-form $\omega \in \wedge^2 \mathfrak{g}^\vee$ defined by $\omega(x, y) = (t_{+1}, [x, y])$, $x, y \in \mathfrak{g}$. The restriction of such form to $\mathfrak{g}_{-1/2}$ is non singular. By the Darboux theorem, there is a direct sum decomposition $\mathfrak{g}_{-1/2} = \mathfrak{p}_{-1/2} \oplus \mathfrak{q}_{-1/2}$ into subspaces of $\mathfrak{g}_{-1/2}$ which are maximally isotropic and dual to each other with respect to ω. Set

$$\mathfrak{r} = \mathfrak{g}_{\leqslant -1} \oplus \mathfrak{p}_{-1/2}\,, \tag{6.4}$$

which is a nilpotent subalgebra of $\mathfrak{g}$. Then, one has

$$\mathsf{X} = \{\Xi \mid \Xi \in \mathrm{Lie\,Gau}_0, \Xi \text{ valued in } \mathfrak{r}\}\,. \tag{6.5}$$

This follows straightforwardly from (5.4b), (4.24), (4.25), the isotropy of $\mathfrak{p}_{-1/2}$ with respect to ω and the gradation of $\mathfrak{g}$ by t_0. From the theory developed in Section 3, it is not difficult to see that the condition of valuedness in $\mathfrak{r}$ is compatible with changes of trivializations of L.

The constraint manifold $\mathsf{W}_{\mathrm{constr}}$ is given in terms of the orthogonal complement $\mathfrak{r}^\perp$ of $\mathfrak{r}$ with respect to the Cartan–Killing form

$$\mathfrak{r}^\perp = \mathfrak{g}_{\leqslant 0} \oplus \mathrm{ad}\, t_{+1}\, \mathfrak{p}_{-1/2}\,, \tag{6.6}$$

and is explicitly given by

$$\mathsf{W}_{\mathrm{constr}} = \{W \mid W \in \mathsf{W},\ W \text{ valued in } \mathfrak{r}^\perp\}\,. \tag{6.7}$$

Here too, one can show that the condition of valuedness in $\mathfrak{r}^\perp$ is compatible with changes of trivializations of L.

From (4.23), (4.24) and (5.7), it follows that, for $u \in \mathrm{Lie\,Conf}_0$ and $W \in \mathsf{W}_{\mathrm{constr}}$, $(\theta_v W)_\kappa \in \mathsf{W}_{\mathrm{constr}}$. Thus the constraints are compatible with the action of Conf_0.

From (4.31) and (5.13), it follows that, for $\Xi \in \mathsf{X}$ and $W \in \mathsf{W}_{\mathrm{constr}}$,

$(\delta_\Xi W)_\kappa \in \mathsf{W}_{\rm constr}$. The gauge symmetry, associated to the first class constraints (6.1), must be fixed. The following can be shown.

Theorem 6.1. *For any $W \in \mathsf{W}_{\rm constr}$, there exists a unique element $\boldsymbol{\Theta}_W \in \mathsf{X}$ depending polynomially on W, R and their derivatives and such that*

$$\mathrm{ad}\, t_{-1}(\exp \boldsymbol{\Theta}_W W)_\kappa = 0 . \tag{6.8}$$

Proof. The proof is quite similar to that of Thm. 3.12. The procedure described by Eqs. (3.26) through (3.30) applies also with Σ replaced by $\Sigma \backslash \Delta$. This leads the construction of $\boldsymbol{\Theta}_W$ by setting $\Omega_0 = W$ and $\exp \boldsymbol{\Theta}_W = \gamma_N \gamma_{N-1} \cdots \gamma_0$. From (6.4) and (6.5), it follows that $\boldsymbol{\Theta}_W \in \mathsf{X}$. The argument explained in Eqs. (3.33) through (3.37) shows also the uniqueness of $\boldsymbol{\Theta}_W$. From (3.5) and (3.27), it appears that $\boldsymbol{\Theta}_W$ depends polynomially on W, R and their derivatives. Note that, unlike in the proof of Thm. 3.12, $\boldsymbol{\Theta}_W$ depends explicitly on R since W is independent of R. □

This theorem generalizes an analogous theorem of Ref. [10]. Here, however, due account is taken of the constraints coming from the global geometry of Σ and L.

Definition 6.1. For any $W \in \mathsf{W}_{\rm constr}$, let

$$W_c = (\exp \boldsymbol{\Theta}_W W)_\kappa . \tag{6.9}$$

From (6.6)–(6.8), W_c belongs to $\mathsf{W}_{\rm constr}$. Clearly, because of the nilpotency of $\mathfrak{r}$, W_c depends polynomially on W, R and derivatives thereof. The uniqueness of $\boldsymbol{\Theta}_W$ ensures further that the map $W \to W_c$ is gauge invariant, i.e., for $\Xi \in \mathsf{X}$ and $W \in \mathsf{W}_{\rm constr}$,

$$(\exp \Xi W)_{\kappa c} = W_c . \tag{6.10}$$

The above suggests the following gauge fixing condition:

$$W = W_c , \quad W \in \mathsf{W}_{\rm red} , \tag{6.11}$$

defining the reduced manifold $\mathsf{W}_{\rm red}$. $\mathsf{W}_{\rm red}$ can be characterized in terms of a set of second class constraints. Let

$$\mathsf{X}' = \{ \Xi \mid \Xi \in \mathrm{Lie}\, \mathrm{Gau}_0, \Xi \text{ valued in } (\ker \mathrm{ad}\, t_{-1})^\perp \} . \tag{6.12}$$

Then, $\mathsf{W}_{\rm red}$ is the submanifold of W determined by

$$J_\Xi \approx 0 , \qquad \Xi \in \mathsf{X}' , \tag{6.13}$$

and is explicitly given by

$$\mathsf{W}_{\rm red} = \{ W \mid W \in \mathsf{W}, W \text{ valued in } \ker \mathrm{ad}\, t_{-1} \} . \tag{6.14}$$

It is readily verified that (6.3) holds with X replaced by X', showing that the reduced manifold is invariant under Conf_0.

$\mathsf{W}_{\mathrm{red}}$ equipped with the Dirac brackets $\{\cdot,\cdot\}^*_\kappa$ supported on the space $\mathfrak{D}(\mathsf{W}_{\mathrm{red}})$ of differential polynomials on $\mathsf{W}_{\mathrm{red}}$ defines the reduced Poisson manifold $(\mathsf{W}_{\mathrm{red}}, \{\cdot,\cdot\}^*_\kappa)$.

7. The Poisson manifold $(\mathsf{W}_{\mathrm{red}}, \{\cdot,\cdot\}^*_\kappa)$ and the classical W algebra

The task now facing one is the computation of the Dirac brackets $\{\cdot,\cdot\}^*_\kappa$ and the study of the properties of $\mathsf{W}_{\mathrm{red}}$. This is the topic of this last section. In due course, a structure of classical W algebra will emerge.

Remark. In this section, Σ, G, S, Δ and ρ are defined as in Section 5.

Any element $W \in \mathsf{W}_{\mathrm{red}}$ is completely characterized by an ordered sequence $(w_\eta)_{\eta\in\Pi}$ with $w_\eta \in \mathsf{KN}_{j_\eta+1}$. Thus, one has the isomorphism

$$\mathsf{W}_{\mathrm{red}} \simeq \bigoplus_{\eta\in\Pi} \mathsf{KN}_{j_\eta+1}\,, \tag{7.1}$$

which expresses the KN content of $\mathsf{W}_{\mathrm{red}}$. In fact, from (6.14), it follows that an element $W \in \mathsf{W}$ belongs to $\mathsf{W}_{\mathrm{red}}$ if and only if W is of the form

$$W = \sum_{\eta\in\Pi} w_\eta t_{\eta,-j_\eta}\,, \tag{7.2}$$

where $w_\eta \in \mathsf{KN}_{j_\eta+1}$.

The dual space $\mathsf{W}^\vee_{\mathrm{red}}$ of $\mathsf{W}_{\mathrm{red}}$ can be defined as the complex vector space of ordered sequences $X = (x_\eta)_{\eta\in\Pi}$ with $x_\eta \in \mathsf{KN}_{-j_\eta}$ with the dual pairing being given by

$$\langle X, W\rangle = \sum_{\eta\in\Pi} N_\eta \langle x_\eta, w_{\bar\eta}\rangle\,, \tag{7.3}$$

see Eq. (2.10). Thus, one has the isomorphism

$$\mathsf{W}^\vee_{\mathrm{red}} \simeq \bigoplus_{\eta\in\Pi} \mathsf{KN}_{-j_\eta}\,. \tag{7.4}$$

Since $\mathsf{W}_{\mathrm{red}}$ is a subspace of W, it is possible to characterize $\mathsf{W}^\vee_{\mathrm{red}}$ as the quotient of $\mathsf{W}^\vee$ by the annihilator of $\mathsf{W}_{\mathrm{red}}$ in $\mathsf{W}^\vee$ under the non singular dual pairing (4.14). The quotient is parametrized by assigning an element of each equivalence class. Of course, this should be done according to a convenient criterion. To this end, the following theorem is useful.

Theorem 7.1. *For any $X \in \mathsf{W}^\vee_{\mathrm{red}}$ and any $V \in \mathsf{W}_{\mathrm{red}}$, there is a unique element $E \in \mathsf{W}^\vee$*

such that

$$E_{\eta, j_\eta} = x_\eta, \quad \eta \in \Pi, \tag{7.5a}$$

$$\mathrm{ad}\, t_{-1}(\partial_A - \mathrm{ad}\, V)E = 0. \tag{7.5b}$$

Explicitly, one has

$$E = [1 + N\, \mathrm{ad}\, t_{-1}(\partial_A - \mathrm{ad}\, V)]^K P_X, \quad P_X = \sum_{\eta \in \Pi} x_\eta t_{\eta, j_\eta}, \tag{7.6}$$

where N is the formal inverse of $M = \frac{1}{2}\, \mathrm{ad}\, t_{-1}\, \mathrm{ad}\, t_{+1}$ extended by 0 *on* $\ker \mathrm{ad}\, t_{+1}$ *and $K \in \mathbb{N}$, $K \geqslant 2j_*$, where j_* is defined in Section 2.*

Proof. The proof given here is inspired by methods developed in Ref. [3]. Let π_+ be the projector on $\ker \mathrm{ad}\, t_{+1}$ along $\mathrm{ran}\, \mathrm{ad}\, t_{-1}$. One has

$$NM = MN = 1 - \pi_+, \tag{7.7}$$

$$[\pi_+, M] = 0, \qquad [\pi_+, N] = 0, \tag{7.8a,b}$$

$$[\mathrm{ad}\, t_0, M] = 0, \qquad [\mathrm{ad}\, t_0, N] = 0. \tag{7.9a,b}$$

Consider Eq. (7.5b). Next, I shall show that it can be solved locally in any coordinate patch and give its general solution. Using (7.7), (7.8a) and (3.21), one checks that (7.5b) is equivalent to

$$\begin{aligned} &[1 - N\, \mathrm{ad}\, t_{-1}(\partial - \mathrm{ad}(V - Rt_{-1}))](1 - \pi_+)E \\ &\quad = N\, \mathrm{ad}\, t_{-1}(\partial - \mathrm{ad}(V - Rt_{-1}))\pi_+ E. \end{aligned} \tag{7.10}$$

The operator $N\, \mathrm{ad}\, t_{-1}(\partial - \mathrm{ad}(V - Rt_{-1}))$ satisfies the relations

$$[N\, \mathrm{ad}\, t_{-1}(\partial - \mathrm{ad}(V - Rt_{-1}))]^K = 0, \qquad K > 2j_*, \tag{7.11a}$$

$$N\, \mathrm{ad}\, t_{-1}(\partial - \mathrm{ad}(V - Rt_{-1})) = N\, \mathrm{ad}\, t_{-1}(\partial_A - \mathrm{ad}\, V) + 1 - \pi_+. \tag{7.11b}$$

Eq. (7.11a) follows from (7.9b) and the fact that $\mathrm{ad}\, t_{-1}$ lowers the degree by 1. Eq. (7.11b) follows from (3.21) and (7.7). Recall that, for a nilpotent operator T, $(1 - T)^{-1}$ is defined and it is given by the series $\sum_{n=0}^{\infty} T^n$ containing only a finite number of non vanishing terms. From (7.11), one has then

$$\begin{aligned} E &= [1 - N\, \mathrm{ad}\, t_{-1}(\partial - \mathrm{ad}(V - Rt_{-1}))]^{-1}\pi_+ E \\ &= \sum_{n=0}^{K} (N\, \mathrm{ad}\, t_{-1}(\partial_A - \mathrm{ad}\, V) + 1 - \pi_+)^n \pi_+ E \\ &= (1 + N\, \mathrm{ad}\, t_{-1}(\partial_A - \mathrm{ad}\, V))^K \pi_+ E, \quad K \geqslant 2j_*. \end{aligned} \tag{7.12}$$

This proves that the local solution of (7.5b) is completely determined by $\pi_+ E$. This suffices to show the local existence and uniqueness of the solution of (7.5). For any patch a, let E_a be a local solution. From (3.1), it is easy to verify that

$$\operatorname{ad} t_{-1}(\partial_{Aa} - \operatorname{ad} V_a)(\operatorname{Ad} L_{ab}E_b - E_a) = 0 . \tag{7.13}$$

By the local uniqueness, it appears that the holomorphic $\mathfrak{g}$-valued 0-cochain $\{E_a\}$ defines an element $E \in W^\vee$ if and only if $\pi_+(\operatorname{Ad} L_{ab}E_b - E_a) = 0$. From (3.3) and (3.4), this condition is equivalent to $E_{\eta, j_\eta} \in \mathsf{KN}_{j_\eta}$ for $\eta \in \Pi$. □

Definition 7.1. For any $X \in \mathsf{W}^\vee_{\mathrm{red}}$ and any $V \in \mathsf{W}_{\mathrm{red}}$, let X_V be the element E of $\mathsf{W}^\vee$ given by (7.6).

For fixed $V \in \mathsf{W}_{\mathrm{red}}$, the map $X \to X_V$ defines a linear injection of $\mathsf{W}^\vee_{\mathrm{red}}$ into $\mathsf{W}^\vee$ with the property that

$$\langle X, W \rangle = \langle X_V, W \rangle , \tag{7.14}$$

for any $X \in \mathsf{W}^\vee_{\mathrm{red}}$ and $W \in \mathsf{W}_{\mathrm{red}}$, where the pairing in the right hand side is the one defined by (4.14). The above relation follows from (2.10), (7.2) and (7.6). Note that $\langle X_V, W \rangle$ is actually independent of V, since only the components $X_{V\eta, j_\eta} = x_\eta$ contribute to the result. This expression for $\langle X, W \rangle$ is important because it can obviously be extended to any $W \in \mathsf{W}$.

The Dirac brackets $\{\cdot, \cdot\}^*_\kappa$ are completely defined by those of the linear functionals

$$\lambda_X(W) = \langle X, W \rangle = \lambda_{X_V}(W) , \qquad W \in \mathsf{W}_{\mathrm{red}} , \tag{7.15}$$

for $X \in \mathsf{W}^\vee_{\mathrm{red}}$, where I have used (7.14) and (5.3) and V is any element of $\mathsf{W}_{\mathrm{red}}$.

The calculation of the Dirac brackets of the λ_X involves the choice of a basis of X'. Luckily the explicit expression of the basis elements is not necessary to carry out the calculation.

Theorem 7.2. *For any $X, Y \in \mathsf{W}^\vee_{\mathrm{red}}$, one has*

$$\begin{aligned} \{\lambda_X, \lambda_Y\}^*_\kappa(W) &= \langle [X_{\kappa^{-1}W}, Y_0], W \rangle + \kappa\chi(X_0, Y_0) , \\ &= \langle [X_0, Y_{\kappa^{-1}W}], W \rangle + \kappa\chi(X_0, Y_0) , \quad W \in \mathsf{W}_{\mathrm{red}} . \end{aligned} \tag{7.16}$$

Proof. From (5.3) and (5.4), for any $\Xi \in \mathrm{Lie\,Gau}_0$ and $V \in \mathsf{W}_{\mathrm{red}}$, one has

$$\{J_\Xi, \lambda_{X_V}\}_\kappa(W) = \kappa \langle \Xi, (\partial_A - \kappa^{-1} \operatorname{ad} W) X_V \rangle , \quad W \in \mathsf{W} . \tag{7.17}$$

From (6.12) and (7.17), it follows that

$$\{J_\Xi, \lambda_{X_V}\}_\kappa(W) |_{V = \kappa^{-1}W} = 0 , \quad \Xi \in \mathsf{X}', W \in \mathsf{W}_{\mathrm{red}} . \tag{7.18}$$

From this relation and the well-known formula of the Dirac brackets, one obtains

$$\begin{aligned} \{\lambda_X, \lambda_Y\}^*_\kappa(W) &= \{\lambda_{X_V}, \lambda_{Y_0}\}_\kappa(W) |_{V = \kappa^{-1}W} \\ &= \{\lambda_{X_0}, \lambda_{Y_V}\}_\kappa(W) |_{V = \kappa^{-1}W} , \quad W \in \mathsf{W}_{\mathrm{red}} . \end{aligned} \tag{7.19}$$

In the second member, I have used the fact that the Dirac bracket is independent of the extension λ_{Y_V} of λ_Y to W used to set $V=0$ and analogously in the third member. The cocycle term is

$$\langle X_{\kappa^{-1}W}, \partial_A Y_0\rangle = -\langle Y_{\kappa^{-1}W}, \partial_A X_0\rangle = \langle X_0, \partial_A Y_0\rangle\ , \quad W\in\mathsf{W}_{\rm red}\ , \tag{7.20}$$

since, by (7.5b), $\partial_A X_0$, $\partial_A Y_0 \in \mathsf{W}_{\rm red}$ and $\langle X_V, W\rangle$ is independent of $V\in\mathsf{W}_{\rm red}$ for any $W\in\mathsf{W}_{\rm red}$. □

The first term in the right hand side of (7.16) is a differential polynomial in the x_η, y_η and w_η and is computable in principle using (7.16). The second term, proportional to κ, is the anomaly. It can be calculated explicitly. The result is

$$\chi(X_0, Y_0) = \sum_{\eta\in\Pi} N_\eta \left[\prod_{m\in I_\eta, m\geq -j_\eta+1} \frac{2}{C^{-1}{}_{j_\eta,m}}\right] \langle x_\eta, D_{j_\eta} y_{\bar\eta}\rangle\ , \tag{7.21}$$

$$\begin{aligned}
&D_0 = \partial\ ,\\
&D_{1/2} = \partial^2 + \tfrac{1}{2}R\ ,\\
&D_1 = \partial^3 + 2R\partial + (\partial R)\ ,\\
&D_{3/2} = \partial^4 + 5R\partial^2 + 5(\partial R)\partial + \tfrac{3}{2}(\partial^2 R + \tfrac{3}{2}R^2)\ ,\\
&D_2 = \partial^5 + 10R\partial^3 + 15(\partial R)\partial^2 + [9(\partial^2 R) + 16R^2]\partial + 2[(\partial^3 R) + 8R(\partial R)]\ ,\\
&\text{etc.}
\end{aligned} \tag{7.22}$$

The D_j are the well-known Bol operators [21].

There are other relevant Dirac brackets. Consider the energy–momentum tensor T. For any $u\in\mathrm{Lie\,Conf}_0$, the restriction of T_u to $\mathsf{W}_{\rm red}$, which will be denoted by the same symbol, is given by (5.9b) with $W\in\mathsf{W}_{\rm red}$. Explicitly,

$$T_u = 2^{-1/2}(t_0, t_0)\langle u, w_o\rangle + (1/2\kappa) \sum_{\eta\in\Pi, j_\eta=0} N_\eta \langle u, w_\eta^{\otimes 2}\rangle\ , \tag{7.23}$$

where $o\in\Pi$ is defined in Section 2. As appears, $T_u \in \mathfrak{D}(\mathsf{W}_{\rm red})$. Note that only the components w_η with $\eta\in\Pi$ and $j_\eta=0$, which correspond to $\ker \mathrm{ad}\, t_{-1} \cap \mathfrak{c}_5$, contribute to the term quadratic in W [2]. From (7.16), one has

$$\{T_u, T_v\}^*_\kappa = T_{[u,v]} + 12\kappa(t_0, t_0)\sigma(u, v)\ , \tag{7.24}$$

for any u, $v\in\mathrm{Lie\,Conf}_0$, which is to be compared with (5.17). Using (7.16), one also finds

$$\{T_u, \lambda_X\}^*_\kappa = \lambda_{\theta_u X} + \kappa\chi(\dot{L}(u), X_0)\ , \tag{7.25}$$

for any $u\in\mathrm{Lie\,Conf}_0$ and any $X\in \mathsf{W}^\vee_{\rm red}$, where $\theta_u X = (\theta_u x_\eta)_{\eta\in\Pi}$ is given by (4.11)

[2] In Ref. [6], such a quadratic contribution was overlooked.

with $\phi = x_\eta$ and $h = -j_\eta$. The calculations involved in deducing (7.24) and (7.25) are straightforward.

Let us discuss briefly the results just obtained. Eq. (7.16) defines a Dirac bracket W algebra in the so-called lowest weight gauge. In fact, analogous expressions have been worked out in the literature following closely related techniques (see Refs. [9–13]). The W algebra proper is obtained by letting x_η and y_η in (7.16) be elements of the KN basis of KN_{-j_η}. The form of the anomaly was first found in Ref. [29] in a different approach, where, however, the deep relation with the theory of SL(2, $\mathbb{C}$) embeddings into simple Lie groups was not apparent. From (7.24), it follows that the T_u form a Dirac bracket Virasoro algebra of classical central charge $12\kappa(t_0, t_0)$. From (7.25), it also appears that the functions λ_X with $x_o = 0$ are primary with respect to the Virasoro algebra. All the above properties have a counterpart in the standard algebraic formulation to W algebras [9–13].

One may consider the W algebra obtained above in terms of modes. For any $\eta \in \Pi$ and any $M \in \mathbb{Z} + p_{j_\eta}$, let $X_{\eta,M}$ be the element of $\mathsf{W}^\vee_{\mathrm{red}}$ defined by the ordered sequence $(\delta_{\eta,\zeta} v_M^{(-j_\eta)})_{\zeta \in \Pi}$. Set

$$j_{\eta,M} = \lambda_{X_{\eta,M}} \,. \tag{7.26}$$

From (7.16), by means of a straightforward calculation, one finds, to order $\mathrm{O}(\kappa^0)$,

$$\begin{aligned}\{j_{\eta,M}, j_{\zeta,N}\}^*_\kappa = {} & \kappa N_\eta \delta_{\eta,\bar{\zeta}} \Bigg[\prod_{m \in I_\eta, m \geqslant -j_\eta+1} \frac{2}{C^{-1}{}_{j_\eta,m}} \Bigg] \zeta_M^{(-j_\eta)}{}_N^{(-j_\eta)} \\ & + \sum_{\xi \in \Pi} \sum_{L \in \mathbb{Z}+p_{j_\xi}} F_{\eta,\zeta}{}^\xi h_M^{(j_\eta)}{}_N^{(j_\zeta)}{}^L_{(1+j_\xi)} j_{\xi,L} + \mathrm{O}(\kappa^{-1}) \,,\end{aligned} \tag{7.27}$$

for $\eta, \zeta \in \Pi$, $M \in \mathbb{Z} + p_{j_\eta}$ and $N \in \mathbb{Z} + p_{j_\zeta}$. Here,

$$\begin{aligned} & h_M^{(j_\eta)}{}_N^{(j_\zeta)}{}^L_{(1+j_\xi)} = \Bigg\langle \sum_{m \in I_\eta, n \in I_\zeta, m+n=j_\xi} (j_\eta, m; j_\zeta, n | j_\xi, j_\xi) X_{\eta,M\,0\,\eta,m} X_{\zeta,N\,0\,\zeta,n}, v_{-L}^{(1+j_\xi)} \Bigg\rangle \,, \\ & \zeta_M^{(-j_\eta)}{}_N^{(-j_\eta)} = \langle v_M^{(-j_\eta)}, D_{j_\eta} v_N^{(-j_\eta)} \rangle \,, \\ & X_{\eta,M\,0} = F_{0,\eta,j_\eta}(v_M^{(-j_\eta)}) \,, \qquad \text{etc.} \end{aligned} \tag{7.28}$$

(cf. Eq. (3.12)) and $F_{\eta,\zeta}{}^\xi$ and N_η are defined in Section 2 (cf. Eqs. (2.3) and (2.10)). It can also be seen that $h_M^{(j_\eta)}{}_N^{(j_\zeta)}{}^L_{(1+j_\xi)}$ and $\zeta_M^{(-j_\eta)}{}_N^{(-j_\eta)}$ vanish unless $j_\eta + j_\zeta - 2j_\xi \leqslant L - M - N \leqslant [2(j_\eta + j_\zeta - j_\xi) + 1]l + 2j_\xi - j_\eta - j_\zeta$ and $-2(2j_\eta + 1)l \leqslant M + N \leqslant 0$, respectively, if the weights involved are non exceptional. The expression of $X_{\eta,M\,0}$ follows easily from noting that the equation $\mathrm{ad}\, t_{-1} \partial_A X_{\eta,M\,0} = 0$ obeyed by $X_{\eta,M\,0}$ is equivalent to (3.5) with $h = 0$, $\mu = j_\eta$ and $\phi = v_N^{(j_\eta)}$.

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